

Probability

MATH 110: General Education Math

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Fall 2026



Probability

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- 2 Probability by Counting
- 3 Mutually Exclusive Events and the Addition Rule
- 4 Independence and the Multiplication Rule
- 5 Conditional Probability
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Sample Spaces and Events

An **experiment** is any process whose result is uncertain, such as tossing a coin, rolling a die, or running a screening test. Each possible result of an experiment is called an **outcome**, or a **sample point**. The set of all sample points is called the **sample space** of the experiment, and it is usually written S .

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Example: Suppose you toss a coin. The coin can land on heads (H) or tails (T), so the sample space is

$$S = \{H, T\}.$$

A screening test is an experiment too: its sample space is $S = \{\text{positive}, \text{negative}\}$.



Sample Spaces and Events

Example: Suppose you toss a coin 2 times. Each time, the coin lands on heads or tails, so the sample space is

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Notice that HT and TH are different sample points: in the first, the heads came first.

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Notice that HT and TH are different sample points: in the first, the heads came first.

Example: If you toss a coin 3 times, the sample space is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

By the Fundamental Principle of Counting, $|S| = 2 \cdot 2 \cdot 2 = 8$. In general, tossing a coin n times gives a sample space with 2^n sample points.

Sample Spaces and Events

A six-sided *die* is a cube with the numbers 1 through 6 on its faces, placed so that opposite faces add up to 7. When the die is rolled, it is said to land on the number that is face up when it stops.



Example: Suppose you roll a die. It can land on any whole number from 1 to 6, so the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Sample Spaces and Events

Example: Suppose you roll two dice. Each sample point is an *ordered pair*: the first number is the number the first die lands on, and the second number is the number the second die lands on. The sample space S consists of the 36 ordered pairs

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

By the Fundamental Principle of Counting, $|S| = 6 \cdot 6 = 36$. As with the coins, order matters: (3, 4) and (4, 3) are different sample points.

Sample Spaces and Events

Every person has one of eight blood types. Red blood cells can carry three markers, called the *A antigen*, the *B antigen*, and the *Rh antigen*. The letters in a blood type record the A and B antigens: type A has only the A antigen, type B has only the B antigen, type AB has both, and type O has neither. The sign records the Rh antigen: a positive type has it, and a negative type does not.

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Example: Choose a person at random from a certain city and record their blood type. The sample space is

$$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}.$$

By the Fundamental Principle of Counting, $|S| = 4 \cdot 2 = 8$. Notice that the experiment involves a person, but the sample points are the blood types. We will return to this experiment throughout the unit.

Sample Spaces and Events

An **event** is a subset of the sample space. We say that an event occurs if the result of the experiment is one of its sample points. Events can be described by listing their sample points, or in words.

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Example: When you roll a die, the event “the die lands on an even number” is $\{2, 4, 6\}$.

Example: When you toss a coin 3 times, the event “at least two heads” is

$$A = \{HHH, HHT, HTH, THH\},$$

and the event “exactly two tails” is

$$B = \{HTT, THT, TTH\}.$$

Earlier we talked about *subsets* of a *universal set*. This is precisely what we have here: the *sample space* is the universal set and its subsets are called *events*.

Sample Spaces and Events

Example: In the blood type experiment, the event “the person has the A antigen” is $\{A+, A-, AB+, AB-\}$, and the event “the person is Rh negative” is $\{A-, B-, AB-, O-\}$.

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Blood types matter for transfusions. A patient’s immune system attacks red blood cells that carry an antigen the patient’s own red blood cells do not have. So a patient with type A+ blood, whose red blood cells have the A and Rh antigens, can safely receive red blood cells only from a donor whose red blood cells have no other antigens. The event “the person can donate red blood cells to a type A+ patient” is

$$\{A+, A-, O+, O-\}.$$

Check Your Understanding

In the blood type experiment, write each event by listing its sample points.

- (a) The person can donate red blood cells to a type B+ patient.
- (b) The person can donate red blood cells to a type AB+ patient.
- (c) The person can donate red blood cells to a type O− patient.

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- (a) The person can donate red blood cells to a type B+ patient.
- (b) The person can donate red blood cells to a type AB+ patient.
- (c) The person can donate red blood cells to a type O− patient.

Solution:

- (a) The patient's cells have the B and Rh antigens, so the donor's cells must not have the A antigen: $\{B+, B-, O+, O-\}$.
- (b) The patient's cells have all three antigens, so any donor will do: the event is all of S .
- (c) The patient's cells have no antigens, so the donor's cells must have none either: $\{O-\}$.

A person with type O− blood can donate red blood cells to anyone, and a person with type AB+ blood can receive red blood cells from anyone.

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Solution: Looking through the 36 sample points for pairs that add up to 5 gives

$$A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

Check Your Understanding

Three patients are each tested for the flu, and each test comes back positive (P) or negative (N). Write the sample space by listing its sample points, and then write the event “exactly one of the patients tests positive.”

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Three patients are each tested for the flu, and each test comes back positive (P) or negative (N). Write the sample space by listing its sample points, and then write the event “exactly one of the patients tests positive.”

Solution: Each sample point records the three results in order, so by the Fundamental Principle of Counting there are $2 \cdot 2 \cdot 2 = 8$ of them:

$$S = \{PPP, PPN, PNP, NPP, PNN, NPN, NNP, NNN\}.$$

The event “exactly one of the patients tests positive” is

$$\{PNN, NPN, NNP\}.$$

Probability by Counting

Consider an experiment with sample space S . To each sample point x in S we assign a number $P(x)$, called the **probability** of x , which measures how likely x is. These numbers must satisfy two conditions:

- 1 $0 \leq P(x) \leq 1$ for every sample point x in S ;
- 2 the probabilities of all of the sample points in S add up to 1.

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A probability of 0 means the sample point is impossible, and a probability of 1 means it is certain. The closer $P(x)$ is to 1, the more likely x is.

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Probabilities can be written as fractions, decimals, or percents. For example,

$$\frac{1}{4} = 0.25 = 25\%.$$

Probability by Counting

Sums with many terms come up so often that there is a compact notation for them, called **sigma notation**.

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This is read “the sum from i equals 1 to n of a_i .”

Probability by Counting

So,

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n.$$

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$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n.$$

Example: In the Counting unit, we found that the number of Zeta monograms is

$$1 + 2 + 3 + \dots + 24 = 300.$$

In sigma notation,

$$\sum_{i=1}^{24} i = 300.$$

Here the summand is $a_i = i$.

Probability by Counting

The summand can be any expression involving the index, as long as that expression makes sense for every value of the index in the sum. For example,

$$\begin{aligned}\sum_{i=2}^7 \frac{1}{i-1} &= \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{5-1} + \frac{1}{6-1} + \frac{1}{7-1} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \\ &= \frac{49}{20}.\end{aligned}$$

Notice that the index does not have to start at 1.

Probability by Counting

You can use any symbol you like for the index. The sums

$$\sum_{i=1}^n a_i, \quad \sum_{k=1}^n a_k, \quad \text{and} \quad \sum_{j=1}^n a_j$$

all mean exactly the same thing.

Check Your Understanding

Evaluate the following sum.

$$\sum_{j=2}^7 (2j - 5)$$

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Solution:

$$\begin{aligned}\sum_{j=2}^7 (2j - 5) &= (2(2) - 5) + (2(3) - 5) + (2(4) - 5) + (2(5) - 5) + (2(6) - 5) + (2(7) - 5) \\ &= (4 - 5) + (6 - 5) + (8 - 5) + (10 - 5) + (12 - 5) + (14 - 5) \\ &= -1 + 1 + 3 + 5 + 7 + 9 \\ &= 24.\end{aligned}$$

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Solution:

$$\begin{aligned}\sum_{m=0}^4 2^m &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 \\ &= 1 + 2 + 4 + 8 + 16 \\ &= 31.\end{aligned}$$

Check Your Understanding

Write each sum in sigma notation.

(a) $3 + 6 + 9 + 12 + \dots + 30$

(b) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{10}$

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(b) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{10}$

Solution:

(a) Each term is 3 times a whole number, from $3 \cdot 1$ up to $3 \cdot 10$, so the sum is $\sum_{i=1}^{10} 3i$.

(b) Each term is 1 divided by a whole number, from 2 up to 10, so the sum is $\sum_{i=2}^{10} \frac{1}{i}$.

Other answers are possible. For example, the sum in part (b) is also $\sum_{i=1}^9 \frac{1}{i+1}$.

Probability by Counting

Sometimes the index of a sum runs through the elements of a set instead of a list of whole numbers. If S is a set, then

$$\sum_{x \in S} P(x)$$

means “add up the numbers $P(x)$ as x runs through the elements of S ,” and it is read “the sum over x in S of $P(x)$.”

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Example: If $S = \{H, T\}$, then $\sum_{x \in S} P(x) = P(H) + P(T)$.

Example: If $S = \{1, 2, 3, 4, 5, 6\}$, then $\sum_{x \in S} P(x) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$.

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Example: If $S = \{1, 2, 3, 4, 5, 6\}$, then $\sum_{x \in S} P(x) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$.

With this notation, the second condition on probabilities becomes

$$\sum_{x \in S} P(x) = 1.$$

Probability by Counting

Example: Suppose it is known that the blood types of the residents of the city in the blood type experiment are distributed as follows:

Blood type	A+	A−	B+	B−	AB+	AB−	O+	O−
Probability	0.32	0.08	0.08	0.02	0.04	0.01	0.36	0.09

Each of these probabilities is between 0 and 1, and

$$\sum_{x \in S} P(x) = 0.32 + 0.08 + 0.08 + 0.02 + 0.04 + 0.01 + 0.36 + 0.09 = 1.$$

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Each of these probabilities is between 0 and 1, and

$$\sum_{x \in S} P(x) = 0.32 + 0.08 + 0.08 + 0.02 + 0.04 + 0.01 + 0.36 + 0.09 = 1.$$

Notice that these eight sample points are not equally likely: a randomly chosen resident is 36 times as likely to have type O+ blood as type AB−.

Probability by Counting

In many experiments, every sample point has the same probability, say p . We then say the sample points are **equally likely**. Tossing a fair coin and rolling a fair die are experiments like this.

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In that case, the second condition tells us what p is:

$$1 = \sum_{x \in S} P(x) = \underbrace{p + p + \dots + p}_{|S| \text{ terms}} = |S| \cdot p,$$

$$\text{so } p = \frac{1}{|S|}.$$

Equally Likely Sample Points

If the sample points in S are equally likely, then $P(x) = \frac{1}{|S|}$ for every sample point x in S .

Probability by Counting

Example: When you toss a fair coin, $S = \{H, T\}$, so $P(H) = P(T) = \frac{1}{2}$.

Example: When you roll a fair die, $S = \{1, 2, 3, 4, 5, 6\}$, so each sample point has probability $\frac{1}{6}$.

Example: When you toss a fair coin n times, there are 2^n sample points, so each one has probability $\frac{1}{2^n}$. For $n = 3$, each of the 8 sample points has probability $\frac{1}{8}$.

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Example: When you roll two fair dice, each of the 36 sample points has probability $\frac{1}{36}$. For instance, $P(3, 4) = \frac{1}{36}$, and separately, $P(4, 3) = \frac{1}{36}$.

Probability by Counting

The **probability** of an event A is the sum of the probabilities of its sample points:

$$P(A) = \sum_{x \in A} P(x).$$

It measures how likely it is that A occurs. Since S itself is an event, the second condition says $P(S) = 1$. Also, $P(\emptyset) = 0$.

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It measures how likely it is that A occurs. Since S itself is an event, the second condition says $P(S) = 1$. Also, $P(\emptyset) = 0$.

Example: When you roll two fair dice, the event “the numbers add up to 5” is $A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, so

$$P(A) = P(1, 4) + P(2, 3) + P(3, 2) + P(4, 1) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}.$$

Probability by Counting

Example: In the blood type experiment, what is the probability that the person has the A antigen? What is the probability that the person can donate red blood cells to a type A+ patient?

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Example: In the blood type experiment, what is the probability that the person has the A antigen? What is the probability that the person can donate red blood cells to a type A+ patient?

Solution: The event “the person has the A antigen” is $\{A+, A-, AB+, AB-\}$, so its probability is

$$P(A+) + P(A-) + P(AB+) + P(AB-) = 0.32 + 0.08 + 0.04 + 0.01 = 0.45.$$

The event “the person can donate red blood cells to a type A+ patient” is $\{A+, A-, O+, O-\}$, so its probability is

$$P(A+) + P(A-) + P(O+) + P(O-) = 0.32 + 0.08 + 0.36 + 0.09 = 0.85.$$

Probability by Counting

When the sample points are equally likely, each one has probability $\frac{1}{|S|}$, so for any event A ,

$$P(A) = \sum_{x \in A} P(x) = \underbrace{\frac{1}{|S|} + \frac{1}{|S|} + \dots + \frac{1}{|S|}}_{|A| \text{ terms}} = |A| \cdot \frac{1}{|S|} = \frac{|A|}{|S|}.$$

Probability by Counting

If the sample points in S are equally likely, then for any event A ,

$$P(A) = \frac{|A|}{|S|} = \frac{\text{number of sample points in } A}{\text{number of sample points in } S}.$$

So when the sample points are equally likely, finding a probability comes down to counting.

Check Your Understanding

A fair coin is tossed 4 times. Write the sample space by listing its 16 sample points. Then write the event “exactly two tails” by listing its sample points, and find its probability.

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Solution: The sample space is

$$S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, \\ THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT\}.$$

The event is $A = \{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$. The sample points are equally likely, so

$$P(A) = \frac{|A|}{|S|} = \frac{6}{16} = \frac{3}{8}.$$

Check Your Understanding

Two fair dice are rolled. Let A be the event “the numbers the dice land on add up to 7,” and let B be the event “both dice land on the same number.” Find $P(A)$ and $P(B)$.

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Two fair dice are rolled. Let A be the event “the numbers the dice land on add up to 7,” and let B be the event “both dice land on the same number.” Find $P(A)$ and $P(B)$.

Solution: Listing the sample points gives

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\},$$

$$B = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\},$$

so $|A| = |B| = 6$. The 36 sample points are equally likely, so

$$P(A) = P(B) = \frac{6}{36} = \frac{1}{6}.$$

Probability by Counting

Be careful to count sample points, and not possible answers. When a fair coin is tossed 3 times, the number of heads can be 0, 1, 2, or 3. These are four possible answers, but they are not equally likely:

Number of heads	Sample points	Probability
0	TTT	$\frac{1}{8}$
1	HTT, THT, TTH	$\frac{3}{8}$
2	HHT, HTH, THH	$\frac{3}{8}$
3	HHH	$\frac{1}{8}$

So the probability of exactly one head is $\frac{3}{8}$, not $\frac{1}{4}$. Notice that the probabilities in the table add up to 1, since every sample point is counted exactly once.

Check Your Understanding

A student reasons: “There are eight blood types, so the probability that a randomly chosen resident of the city has type AB— blood is $\frac{1}{8}$.” What is wrong with this reasoning?

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Solution: Probability by counting, $P(E) = \frac{|E|}{|S|}$, only works when the sample points are equally likely, and the eight blood types are not. In this city, the probability that a randomly chosen resident has type AB– blood is $P(\text{AB–}) = 0.01$, far less than $\frac{1}{8} = 0.125$.

Probability by Counting

Here is another experiment whose sample points are not equally likely. Consider a coin that is weighted so that it lands on heads with probability $\frac{2}{3}$ and on tails with probability $\frac{1}{3}$. If you toss it twice, the sample space is still $S = \{HH, HT, TH, TT\}$, but now

$$P(HH) = \frac{4}{9}, \quad P(HT) = P(TH) = \frac{2}{9}, \quad \text{and} \quad P(TT) = \frac{1}{9}.$$

We will see where these numbers come from in the section on independence.

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We will see where these numbers come from in the section on independence.

Probability by counting no longer works, but we can still add up the probabilities of the sample points. The event “at least one head” is $A = \{HH, HT, TH\}$, so

$$P(A) = P(HH) + P(HT) + P(TH) = \frac{4}{9} + \frac{2}{9} + \frac{2}{9} = \frac{8}{9}.$$

Check Your Understanding

Using the weighted coin from the previous slide, tossed twice, find the probability of the event “exactly one head.” Then check that the probabilities of the four sample points add up to 1.

Check Your Understanding

Using the weighted coin from the previous slide, tossed twice, find the probability of the event “exactly one head.” Then check that the probabilities of the four sample points add up to 1.

Solution: The event is $C = \{HT, TH\}$, so

$$P(C) = P(HT) + P(TH) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}.$$

Also,

$$\sum_{x \in S} P(x) = \frac{4}{9} + \frac{2}{9} + \frac{2}{9} + \frac{1}{9} = \frac{9}{9} = 1.$$

Mutually Exclusive Events and the Addition Rule

Two events A and B are **mutually exclusive** if they are disjoint, that is, if $A \cap B = \emptyset$. In words, two events are mutually exclusive if they cannot both occur.

Mutually Exclusive Events and the Addition Rule

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Example: Roll a die. The events $\{1, 2\}$ and $\{5, 6\}$ are mutually exclusive, since the die cannot land on a number in both sets.

Mutually Exclusive Events and the Addition Rule

Two events A and B are **mutually exclusive** if they are disjoint, that is, if $A \cap B = \emptyset$. In words, two events are mutually exclusive if they cannot both occur.

Example: Roll a die. The events $\{1, 2\}$ and $\{5, 6\}$ are mutually exclusive, since the die cannot land on a number in both sets.

Example: Roll a die. The events “the die lands on an even number” and “the die lands on a number greater than 3” are not mutually exclusive, since

$$\{2, 4, 6\} \cap \{4, 5, 6\} = \{4, 6\} \neq \emptyset.$$

If the die lands on 4 or 6, both events occur.

Mutually Exclusive Events and the Addition Rule

If A and B are mutually exclusive, every sample point of $A \cup B$ is in exactly one of A and B . So when we add up the probabilities of the sample points in $A \cup B$, we can add up those in A , then those in B , and no sample point is counted twice:

$$P(A \cup B) = \sum_{x \in A \cup B} P(x) = \sum_{x \in A} P(x) + \sum_{x \in B} P(x) = P(A) + P(B).$$

The Addition Rule

If $A_1, A_2, \dots, A_k$ are events and no two of them have any sample points in common, then

$$P(A_1 \cup A_2 \cup \dots \cup A_k) = P(A_1) + P(A_2) + \dots + P(A_k).$$

This is called **the Addition Rule**. It is the Addition Principle from the Counting unit, with probabilities in place of counts.

Mutually Exclusive Events and the Addition Rule

Example: A fair coin is tossed 3 times. What is the probability of getting at least 2 heads?

Mutually Exclusive Events and the Addition Rule

Example: A fair coin is tossed 3 times. What is the probability of getting at least 2 heads?

Solution: Let A be the event “exactly 2 heads” and B the event “exactly 3 heads,” so that

$$A = \{HHT, HTH, THH\} \quad \text{and} \quad B = \{HHH\}.$$

The event “at least 2 heads” is $A \cup B$. The events A and B are mutually exclusive, since no sequence of tosses has both exactly 2 and exactly 3 heads. Using the table from the previous section and the Addition Rule,

$$P(A \cup B) = P(A) + P(B) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}.$$

Check Your Understanding

A fair coin is tossed 4 times. Let A be the event “exactly two heads,” and let B be the event “exactly three heads.” Explain why A and B are mutually exclusive. Then find $P(A)$, $P(B)$, and $P(A \cup B)$.

Check Your Understanding

A fair coin is tossed 4 times. Let A be the event “exactly two heads,” and let B be the event “exactly three heads.” Explain why A and B are mutually exclusive. Then find $P(A)$, $P(B)$, and $P(A \cup B)$.

Solution: No sequence of four tosses has both exactly two heads and exactly three heads, so A and B have no sample points in common. Listing their sample points,

$$A = \{\text{HHTT}, \text{HTHT}, \text{HTTH}, \text{THHT}, \text{THTH}, \text{TTHH}\},$$

$$B = \{\text{HHHT}, \text{HHTH}, \text{HTHH}, \text{THHH}\}.$$

There are 16 equally likely sample points, so $P(A) = \frac{6}{16} = \frac{3}{8}$ and $P(B) = \frac{4}{16} = \frac{1}{4}$. By the Addition Rule,

$$P(A \cup B) = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}.$$

Mutually Exclusive Events and the Addition Rule

For an event A in a sample space S , the complement A^c is the event consisting of the sample points of S that are not in A . It occurs exactly when A does not occur.

Mutually Exclusive Events and the Addition Rule

For an event A in a sample space S , the complement A^c is the event consisting of the sample points of S that are not in A . It occurs exactly when A does not occur.

Since A and A^c are mutually exclusive and $A \cup A^c = S$, the Addition Rule gives

$$P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1.$$

Probabilities of Complements

For any event A in a sample space S ,

$$P(A^c) = 1 - P(A).$$

As with counting, this helps when the complement is easier to work with, as often happens with *at least one*.

Check Your Understanding

In the blood type experiment, use complements to find the probability that the person cannot donate red blood cells to a type A+ patient. Then check your answer by listing the sample points of the event.

Check Your Understanding

In the blood type experiment, use complements to find the probability that the person cannot donate red blood cells to a type A+ patient. Then check your answer by listing the sample points of the event.

Solution: We found that the probability that the person can donate red blood cells to a type A+ patient is 0.85. The event “the person cannot donate red blood cells to a type A+ patient” is its complement, so its probability is

$$1 - 0.85 = 0.15.$$

Check: the complement of $\{A+, A-, O+, O-\}$ is $\{B+, B-, AB+, AB-\}$, and

$$P(B+) + P(B-) + P(AB+) + P(AB-) = 0.08 + 0.02 + 0.04 + 0.01 = 0.15.$$

Check Your Understanding

A fair coin is tossed 10 times. What is the probability of getting at least one head?

Check Your Understanding

A fair coin is tossed 10 times. What is the probability of getting at least one head?

Solution: The sample space has $2^{10} = 1,024$ equally likely sample points, far too many to list. But the complement of “at least one head” is “no heads,” which is the event $A = \{\text{TTTTTTTTTT}\}$ with just one sample point. So the event “at least one head” is A^c , and

$$P(A^c) = 1 - P(A) = 1 - \frac{1}{1,024} = \frac{1,023}{1,024} \approx 0.999.$$

If you toss a fair coin 10 times, it is almost certain that you will get at least one head.

Check Your Understanding

A fair coin is tossed 10 times. What is the probability of getting at least three heads?

Check Your Understanding

A fair coin is tossed 10 times. What is the probability of getting at least three heads?

Solution: Let A be the event “at most two heads,” so that A^c is the event “at least three heads.” A sequence with exactly k heads is determined by choosing which k of the 10 tosses are heads, so there are ${}_{10}C_0 = 1$ sequence with no heads, ${}_{10}C_1 = 10$ with exactly one head, and ${}_{10}C_2 = 45$ with exactly two heads. These cases do not overlap, so by the Addition Principle,

$$|A| = 1 + 10 + 45 = 56.$$

Therefore,

$$P(A^c) = 1 - P(A) = 1 - \frac{56}{1,024} = \frac{968}{1,024} = \frac{121}{128} \approx 0.945.$$

Check Your Understanding

Two fair dice are rolled. What is the probability that at least one of the dice lands on 6?

Check Your Understanding

Two fair dice are rolled. What is the probability that at least one of the dice lands on 6?

Solution: Let A be the event “neither die lands on 6,” so that A^c is the event “at least one die lands on 6.” Each die has 5 numbers other than 6 that it can land on, so by the Fundamental Principle of Counting, $|A| = 5 \cdot 5 = 25$. Therefore,

$$P(A^c) = 1 - P(A) = 1 - \frac{25}{36} = \frac{11}{36}.$$

Mutually Exclusive Events and the Addition Rule

What if the events are not mutually exclusive? Toss a fair coin twice, so that $S = \{HH, HT, TH, TT\}$ and each sample point has probability $\frac{1}{4}$. Let A be the event “at least one head” and B the event “at least one tail.” Find $P(A)$, $P(B)$, $P(A \cup B)$, and $P(A \cap B)$.

Mutually Exclusive Events and the Addition Rule

What if the events are not mutually exclusive? Toss a fair coin twice, so that $S = \{HH, HT, TH, TT\}$ and each sample point has probability $\frac{1}{4}$. Let A be the event “at least one head” and B the event “at least one tail.” Find $P(A)$, $P(B)$, $P(A \cup B)$, and $P(A \cap B)$.

Solution: We have

$$A = \{HH, HT, TH\}, \quad B = \{HT, TH, TT\}, \quad A \cup B = S, \quad A \cap B = \{HT, TH\},$$

so

$$P(A) = \frac{3}{4}, \quad P(B) = \frac{3}{4}, \quad P(A \cup B) = 1, \quad P(A \cap B) = \frac{2}{4} = \frac{1}{2}.$$

Mutually Exclusive Events and the Addition Rule

What if the events are not mutually exclusive? Toss a fair coin twice, so that $S = \{HH, HT, TH, TT\}$ and each sample point has probability $\frac{1}{4}$. Let A be the event “at least one head” and B the event “at least one tail.” Find $P(A)$, $P(B)$, $P(A \cup B)$, and $P(A \cap B)$.

Solution: We have

$$A = \{HH, HT, TH\}, \quad B = \{HT, TH, TT\}, \quad A \cup B = S, \quad A \cap B = \{HT, TH\},$$

so

$$P(A) = \frac{3}{4}, \quad P(B) = \frac{3}{4}, \quad P(A \cup B) = 1, \quad P(A \cap B) = \frac{2}{4} = \frac{1}{2}.$$

Notice that $P(A) + P(B) = \frac{3}{2}$, which is not $P(A \cup B)$. Do you see how these four numbers are related?

Mutually Exclusive Events and the Addition Rule

What if the events are not mutually exclusive? Toss a fair coin twice, so that $S = \{HH, HT, TH, TT\}$ and each sample point has probability $\frac{1}{4}$. Let A be the event “at least one head” and B the event “at least one tail.” Find $P(A)$, $P(B)$, $P(A \cup B)$, and $P(A \cap B)$.

Solution: We have

$$A = \{HH, HT, TH\}, \quad B = \{HT, TH, TT\}, \quad A \cup B = S, \quad A \cap B = \{HT, TH\},$$

so

$$P(A) = \frac{3}{4}, \quad P(B) = \frac{3}{4}, \quad P(A \cup B) = 1, \quad P(A \cap B) = \frac{2}{4} = \frac{1}{2}.$$

Notice that $P(A) + P(B) = \frac{3}{2}$, which is not $P(A \cup B)$. Do you see how these four numbers are related?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Look familiar?

Mutually Exclusive Events and the Addition Rule

When the sample points are equally likely, the formula from the previous slide follows directly from the Inclusion-Exclusion Principle from the Counting unit:

$$\begin{aligned}P(A \cup B) &= \frac{|A \cup B|}{|S|} \\&= \frac{|A| + |B| - |A \cap B|}{|S|} \\&= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} \\&= P(A) + P(B) - P(A \cap B).\end{aligned}$$

Mutually Exclusive Events and the Addition Rule

The same is true when the sample points are not equally likely: adding $P(A)$ and $P(B)$ counts the probabilities of each sample point in $A \cap B$ twice. So,

$$\begin{aligned} P(A \cup B) &= \sum_{x \in A \cup B} P(x) \\ &= \sum_{x \in A} P(x) + \sum_{x \in B} P(x) - \sum_{x \in A \cap B} P(x) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

Mutually Exclusive Events and the Addition Rule

The Inclusion-Exclusion Principle for Events

If A and B are events in a sample space S , then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then $A \cap B = \emptyset$, so $P(A \cap B) = 0$ and this becomes the Addition Rule.

Check Your Understanding

Two fair dice are rolled. Let A be the event “the first die lands on a number less than or equal to the number on the second die,” and let B be the event “the first die lands on a number greater than or equal to the number on the second die.” Find $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A \cup B)$, and check that the Inclusion-Exclusion Principle for Events holds.

Check Your Understanding

Two fair dice are rolled. Let A be the event “the first die lands on a number less than or equal to the number on the second die,” and let B be the event “the first die lands on a number greater than or equal to the number on the second die.” Find $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A \cup B)$, and check that the Inclusion-Exclusion Principle for Events holds.

Solution: Counting in the grid of 36 sample points, $|A| = |B| = 21$. The event $A \cap B$ is “both dice land on the same number,” with 6 sample points, and every sample point is in A or in B , so $A \cup B = S$. Therefore,

$$P(A) = P(B) = \frac{21}{36} = \frac{7}{12}, \quad P(A \cap B) = \frac{6}{36} = \frac{1}{6}, \quad P(A \cup B) = 1,$$

and indeed

$$P(A) + P(B) - P(A \cap B) = \frac{7}{12} + \frac{7}{12} - \frac{2}{12} = \frac{12}{12} = 1 = P(A \cup B).$$

Check Your Understanding

In the blood type experiment, let X be the event “the person has the A antigen” and Y the event “the person has the B antigen.” Find $P(X)$, $P(Y)$, and $P(X \cap Y)$, and use the Inclusion-Exclusion Principle for Events to find $P(X \cup Y)$. Then check your answer using complements.

Check Your Understanding

In the blood type experiment, let X be the event “the person has the A antigen” and Y the event “the person has the B antigen.” Find $P(X)$, $P(Y)$, and $P(X \cap Y)$, and use the Inclusion-Exclusion Principle for Events to find $P(X \cup Y)$. Then check your answer using complements.

Solution: We found that $P(X) = 0.45$. Also, $Y = \{B+, B-, AB+, AB-\}$ and $X \cap Y = \{AB+, AB-\}$, so

$$P(Y) = 0.08 + 0.02 + 0.04 + 0.01 = 0.15 \quad \text{and} \quad P(X \cap Y) = 0.04 + 0.01 = 0.05.$$

By the Inclusion-Exclusion Principle for Events,

$$P(X \cup Y) = 0.45 + 0.15 - 0.05 = 0.55.$$

Check: $X \cup Y$ is the event “the person has the A antigen or the B antigen,” whose complement is $\{O+, O-\}$. So $P(X \cup Y) = 1 - (0.36 + 0.09) = 1 - 0.45 = 0.55$.

Independence and the Multiplication Rule

Toss a fair coin twice, so that $S = \{HH, HT, TH, TT\}$. Let A be the event “the first toss lands on heads” and B the event “the second toss lands on heads.” Then

$$A = \{HH, HT\}, \quad B = \{HH, TH\}, \quad A \cap B = \{HH\},$$

so $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{4}$.

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$$A = \{HH, HT\}, \quad B = \{HH, TH\}, \quad A \cap B = \{HH\},$$

so $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{4}$.

Do you notice anything?

Independence and the Multiplication Rule

Toss a fair coin twice, so that $S = \{HH, HT, TH, TT\}$. Let A be the event “the first toss lands on heads” and B the event “the second toss lands on heads.” Then

$$A = \{HH, HT\}, \quad B = \{HH, TH\}, \quad A \cap B = \{HH\},$$

so $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{4}$.

Do you notice anything?

Notice that $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$. Should the result of the first toss have any influence on the result of the second toss?

Independence and the Multiplication Rule

Mathematically, two events A and B are called **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

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Intuitively, two events are independent if knowing that one of them occurs does not change the probability of the other. We will see why the definition does actually capture the intuitive idea later on in this unit.

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In practice, we often know from the situation that two events cannot influence each other, such as two tosses of a coin, or the test results of two unrelated patients. In that case we assume that the events are independent, and the definition tells us how to find the probability that both occur: multiply their probabilities. Using independence in this way is called **the Multiplication Rule**.

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The same goes for three or more events that cannot influence each other: the probability that all of them occur is the product of their probabilities.

Independence and the Multiplication Rule

Example: A bag contains 1 red marble, 1 yellow marble, and 1 blue marble. A marble is drawn at random, its color is recorded, and it is put back. Then a second marble is drawn and its color is recorded. The sample space is

$$S = \{RR, RY, RB, YR, YY, YB, BR, BY, BB\},$$

and the sample points are equally likely. Let X be the event “the first marble is red” and Y the event “the second marble is yellow or blue.” Are X and Y independent?

Independence and the Multiplication Rule

Example: A bag contains 1 red marble, 1 yellow marble, and 1 blue marble. A marble is drawn at random, its color is recorded, and it is put back. Then a second marble is drawn and its color is recorded. The sample space is

$$S = \{RR, RY, RB, YR, YY, YB, BR, BY, BB\},$$

and the sample points are equally likely. Let X be the event “the first marble is red” and Y the event “the second marble is yellow or blue.” Are X and Y independent?

Solution: The first marble goes back in the bag, so it should not affect the second draw. Thus, we expect X and Y to be independent. Indeed, $X = \{RR, RY, RB\}$, $Y = \{RY, RB, YY, YB, BY, BB\}$, and $X \cap Y = \{RY, RB\}$, so

$$P(X) \cdot P(Y) = \frac{3}{9} \cdot \frac{6}{9} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} = P(X \cap Y),$$

and X and Y are independent.

Independence and the Multiplication Rule

Example: Now suppose the first marble is not put back in the bag before the second marble is drawn. The sample space is

$$S = \{RY, RB, YR, YB, BR, BY\},$$

and again the sample points are equally likely. With X and Y as before, are X and Y independent?

Independence and the Multiplication Rule

Example: Now suppose the first marble is not put back in the bag before the second marble is drawn. The sample space is

$$S = \{RY, RB, YR, YB, BR, BY\},$$

and again the sample points are equally likely. With X and Y as before, are X and Y independent?

Solution: Now $X = \{RY, RB\}$, $Y = \{RY, RB, YB, BY\}$, and $X \cap Y = \{RY, RB\}$, so

$$P(X) \cdot P(Y) = \frac{2}{6} \cdot \frac{4}{6} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \neq \frac{1}{3} = \frac{2}{6} = P(X \cap Y),$$

and X and Y are not independent.

Independence and the Multiplication Rule

Example: Now suppose the first marble is not put back in the bag before the second marble is drawn. The sample space is

$$S = \{RY, RB, YR, YB, BR, BY\},$$

and again the sample points are equally likely. With X and Y as before, are X and Y independent?

Solution: Now $X = \{RY, RB\}$, $Y = \{RY, RB, YB, BY\}$, and $X \cap Y = \{RY, RB\}$, so

$$P(X) \cdot P(Y) = \frac{2}{6} \cdot \frac{4}{6} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \neq \frac{1}{3} = \frac{2}{6} = P(X \cap Y),$$

and X and Y are not independent.

This makes sense: if we know the first marble is red, then only the yellow and blue marbles are left in the bag, so the second marble is certain to be yellow or blue.

Check Your Understanding

You toss a fair coin and roll a fair die. Write the sample space by listing its sample points. Let H be the event “the coin lands on heads” and X the event “the die lands on 6.” Find $P(H)$, $P(X)$, and $P(H \cap X)$. Are H and X independent?

Check Your Understanding

You toss a fair coin and roll a fair die. Write the sample space by listing its sample points. Let H be the event “the coin lands on heads” and X the event “the die lands on 6.” Find $P(H)$, $P(X)$, and $P(H \cap X)$. Are H and X independent?

Solution: By the Fundamental Principle of Counting there are $2 \cdot 6 = 12$ equally likely sample points:

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}.$$

Then H has 6 sample points, $X = \{H6, T6\}$, and $H \cap X = \{H6\}$, so

$$P(H) = \frac{6}{12} = \frac{1}{2}, \quad P(X) = \frac{2}{12} = \frac{1}{6}, \quad P(H \cap X) = \frac{1}{12}.$$

Since $P(H) \cdot P(X) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12} = P(H \cap X)$, the events H and X are independent.

Independence and the Multiplication Rule

Often, independence is *given*: the way the experiment is carried out makes it clear that one part of it cannot influence another. For example, the results are independent when

- tossing a coin several times, since the coin has no memory of earlier tosses;
- rolling several dice, since no die affects how another one lands;
- drawing several times with replacement, since the bag is the same before every draw.

In situations like these, we do not need to check that $P(A \cap B) = P(A) \cdot P(B)$. We assume that the events are independent and use the Multiplication Rule.

Independence and the Multiplication Rule

Now we can see where the probabilities for the weighted coin came from. The coin lands on heads with probability $\frac{2}{3}$ and on tails with probability $\frac{1}{3}$, and the second toss is independent of the first. By the Multiplication Rule,

$$\begin{aligned} P(HH) &= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}, & P(HT) &= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}, \\ P(TH) &= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}, & P(TT) &= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}. \end{aligned}$$

Independence and the Multiplication Rule

Now we can see where the probabilities for the weighted coin came from. The coin lands on heads with probability $\frac{2}{3}$ and on tails with probability $\frac{1}{3}$, and the second toss is independent of the first. By the Multiplication Rule,

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The same reasoning applies to a fair coin. Each toss lands on heads or tails with probability $\frac{1}{2}$, so any particular sequence of n tosses has probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \dots \cdot \frac{1}{2} = \frac{1}{2^n}$, with n factors of $\frac{1}{2}$. This is exactly what we found by counting.

Independence and the Multiplication Rule

So, the Multiplication Rule works even when the sample points aren't equally likely.

Example: A treatment works for 80% of the patients who receive it. Two unrelated patients receive the treatment. What is the probability that both respond? That neither responds? That at least one responds?

Independence and the Multiplication Rule

So, the Multiplication Rule works even when the sample points aren't equally likely.

Example: A treatment works for 80% of the patients who receive it. Two unrelated patients receive the treatment. What is the probability that both respond? That neither responds? That at least one responds?

Solution: The patients are unrelated, so we treat their responses as independent. Writing R for a patient who responds and N for one who does not, the sample space is $\{RR, RN, NR, NN\}$. By the Multiplication Rule,

$$P(RR) = 0.8 \cdot 0.8 = 0.64 \quad \text{and} \quad P(NN) = 0.2 \cdot 0.2 = 0.04.$$

Let E be the event “at least one responds.” Then $E^c = \{NN\}$, so

$$P(E) = 1 - P(NN) = 1 - 0.04 = 0.96.$$

Check Your Understanding

Two unrelated residents of the city in the blood type experiment donate blood, and their blood types are independent. What is the probability that both have type O— blood? What is the probability that at least one of them does?

Check Your Understanding

Two unrelated residents of the city in the blood type experiment donate blood, and their blood types are independent. What is the probability that both have type O− blood? What is the probability that at least one of them does?

Solution: The sample space consists of the $8 \cdot 8 = 64$ ordered pairs of blood types, such as $(A+, O-)$, and they are not equally likely. By the Multiplication Rule,

$$P(O-, O-) = 0.09 \cdot 0.09 = 0.0081.$$

Let E be the event “at least one of them has type O− blood.” Then E^c is the event “neither has type O− blood.” Each donor does not have type O− blood with probability $1 - 0.09 = 0.91$, so $P(E^c) = 0.91 \cdot 0.91 = 0.8281$. By complements,

$$P(E) = 1 - P(E^c) = 1 - 0.8281 = 0.1719.$$

Check Your Understanding

A bag contains 4 red marbles, 7 yellow marbles, and 5 blue marbles. Three marbles are drawn from the bag, and each marble is put back in the bag before the next one is drawn. What is the probability of getting 2 red marbles and 1 yellow marble?

Check Your Understanding

A bag contains 4 red marbles, 7 yellow marbles, and 5 blue marbles. Three marbles are drawn from the bag, and each marble is put back in the bag before the next one is drawn. What is the probability of getting 2 red marbles and 1 yellow marble?

Solution: The event is $A = \{RRY, RYR, YRR\}$. Each marble is put back, so the draws are independent, with $P(R) = \frac{4}{16} = \frac{1}{4}$ and $P(Y) = \frac{7}{16}$ on every draw. By the Multiplication Rule, each of the three sample points in A has probability

$$\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{7}{16} = \frac{7}{256}.$$

These sample points are mutually exclusive, so by the Addition Rule,

$$P(A) = \frac{7}{256} + \frac{7}{256} + \frac{7}{256} = \frac{21}{256} \approx 0.082.$$

Check Your Understanding

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A screening test gives a false positive for 5% of healthy patients. Three healthy patients are tested, and their results are independent. What is the probability that at least one of them gets a false positive?

Solution: Write each result as P (positive) or N (negative). Each healthy patient correctly tests negative with probability $1 - 0.05 = 0.95$, so by the Multiplication Rule,

$$P(\text{NNN}) = 0.95 \cdot 0.95 \cdot 0.95 = 0.95^3 \approx 0.857.$$

Let E be the event “at least one false positive.” Then $E^c = \{\text{NNN}\}$, so

$$P(E) = 1 - P(\text{NNN}) \approx 1 - 0.857 = 0.143.$$

Even a fairly accurate test produces a false positive about 14% of the time when three healthy patients are tested.

Conditional Probability

Let A and B be events in a sample space S . Intuitively, the conditional probability of A given B is the probability that A occurs if we know that B occurs.

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This is the everyday “of those who...” calculation. Suppose 60% of the students in a course pass the final exam, and 15% of the students get an A. Then of those who passed, the fraction who got an A is

$$\frac{15\%}{60\%} = \frac{0.15}{0.60} = 0.25 = 25\%.$$

We divide by 60% because the students who passed are the new “everyone.”

Conditional Probability

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Let A and B be events in a sample space S with $P(B) \neq 0$. The conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

The number $P(A \mid B)$ is called the **conditional probability** of A given B , and it is read “the probability of A given B .” Dividing by $P(B)$ rescales the probabilities of the sample points in B so that, in the new sample space B , they add up to 1.

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When the sample points of S are equally likely, the formula becomes a ratio of counts:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|/|S|}{|B|/|S|} = \frac{|A \cap B|}{|B|}.$$

Conditional Probability

Example: Two fair dice are rolled. Let A be the event “the first die lands on 1, 2, or 3” and B the event “the numbers the dice land on add up to 7.” Find $P(A)$, $P(B)$, $P(A \mid B)$, and $P(B \mid A)$.

Conditional Probability

Example: Two fair dice are rolled. Let A be the event “the first die lands on 1, 2, or 3” and B the event “the numbers the dice land on add up to 7.” Find $P(A)$, $P(B)$, $P(A | B)$, and $P(B | A)$.

Solution: The first die lands on 1, 2, or 3 in $3 \cdot 6 = 18$ of the 36 sample points, so $P(A) = \frac{18}{36} = \frac{1}{2}$. We saw earlier that $P(B) = \frac{6}{36} = \frac{1}{6}$. Also, $A \cap B = \{(1, 6), (2, 5), (3, 4)\}$, so $P(A \cap B) = \frac{3}{36} = \frac{1}{12}$. Therefore,

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad P(B | A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/2} = \frac{1}{6}.$$

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Notice that $P(A | B) = P(A)$ and $P(B | A) = P(B)$: knowing that one of these events occurs does not change the probability of the other. Also, $P(A \cap B) = \frac{1}{12} = \frac{1}{2} \cdot \frac{1}{6} = P(A) \cdot P(B)$, so A and B are independent.

Conditional Probability

The example is no accident. Intuitively, A and B are independent if knowing that B occurs does not change the probability of A , that is, if $P(A \mid B) = P(A)$. In the section on independence, we instead defined independence by $P(A \cap B) = P(A) \cdot P(B)$.

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These say the same thing, as long as $P(A)$ and $P(B)$ are not 0:

$$P(A \mid B) = P(A) \iff \frac{P(A \cap B)}{P(B)} = P(A) \iff P(A \cap B) = P(A) \cdot P(B).$$

In the same way, $P(B \mid A) = P(B)$ says the same thing too.

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In the same way, $P(B \mid A) = P(B)$ says the same thing too.

For the marbles drawn without putting the first one back, knowing that the first marble is red changes the probability that the second is yellow or blue from $\frac{2}{3}$ to 1:

$$P(Y \mid X) = \frac{P(X \cap Y)}{P(X)} = \frac{1/3}{1/3} = 1 \neq \frac{2}{3} = P(Y).$$

Conditional Probability

Example: In the blood type experiment, let X be the event “the person has the A antigen” and Y the event “the person has the B antigen.” If we know that the person has the A antigen, what is the probability that the person has type AB blood? Are X and Y independent?

Conditional Probability

Example: In the blood type experiment, let X be the event “the person has the A antigen” and Y the event “the person has the B antigen.” If we know that the person has the A antigen, what is the probability that the person has type AB blood? Are X and Y independent?

Solution: We found that $P(X) = 0.45$. Let C be the event “type AB,” so $C = \{AB+, AB-\}$ and $P(C) = 0.04 + 0.01 = 0.05$. Every person with type AB blood has the A antigen, so $C \cap X = C$, and $P(C | X) = \frac{P(C)}{P(X)} = \frac{0.05}{0.45} \approx 0.111$. Knowing that the person has the A antigen more than doubles the probability of type AB, from 0.05 to about 0.111.

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Since $X \cap Y = C$, this also shows $P(Y | X) \approx 0.111 \neq 0.15 = P(Y)$. Equivalently, $P(X \cap Y) = 0.05 \neq 0.0675 = P(X) \cdot P(Y)$. So X and Y are not independent: knowing that a person has the A antigen makes the B antigen less likely.

Conditional Probability

It is easy to confuse *mutually exclusive* and *independent*, since both seem to say that two events “have nothing to do with each other.” But they say very different things. Mutually exclusive is a statement about sets: $A \cap B = \emptyset$, so the events cannot both occur. Independent is a statement about probabilities: knowing that one event occurs does not change the probability of the other.

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In fact, suppose A and B are mutually exclusive, with $P(A) \neq 0$ and $P(B) \neq 0$. If B occurs, then A cannot occur, so

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0 \neq P(A).$$

Knowing that B occurs changes the probability of A all the way down to 0. Mutually exclusive events are about as far from independent as two events can be.

Conditional Probability

Example: Roll a fair die. Let A be the event “the die lands on an even number,” B the event “the die lands on an odd number,” and C the event “the die lands on a number less than or equal to 4.”

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$$P(A \mid B) = 0 \neq \frac{1}{2} = P(A).$$

Conditional Probability

Example: Roll a fair die. Let A be the event “the die lands on an even number,” B the event “the die lands on an odd number,” and C the event “the die lands on a number less than or equal to 4.”

The events A and B are mutually exclusive, since $A \cap B = \emptyset$. They are not independent:

$$P(A \mid B) = 0 \neq \frac{1}{2} = P(A).$$

The events A and C are not mutually exclusive, since $A \cap C = \{2, 4\}$. But they are independent:

$$P(A \cap C) = \frac{2}{6} = \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3} = P(A) \cdot P(C).$$

Intuitively, half of the sample points in C are even, the same fraction as in the whole sample space, so learning that C occurs does not change the probability that the number is even.

Check Your Understanding

Can two events A and B with $P(A) \neq 0$ and $P(B) \neq 0$ be both mutually exclusive and independent?

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Can two events A and B with $P(A) \neq 0$ and $P(B) \neq 0$ be both mutually exclusive and independent?

Solution: No. If A and B are mutually exclusive, then

$$P(A \cap B) = P(\emptyset) = 0.$$

If A and B are independent, then

$$P(A \cap B) = P(A) \cdot P(B),$$

which is not 0, since neither $P(A)$ nor $P(B)$ is 0. Both cannot be true at once.

Conditional Probability

A **two-way table** sorts a group by two characteristics at once: one characteristic for the rows, and the other for the columns. Each member of the group is counted in exactly one cell.

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Example: A clinic screens 100 patients for a certain condition using a rapid test:

	Test positive	Test negative	Total
Condition present	18	2	20
Condition absent	8	72	80
Total	26	74	100

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Example: A clinic screens 100 patients for a certain condition using a rapid test:

	Test positive	Test negative	Total
Condition present	18	2	20
Condition absent	8	72	80
Total	26	74	100

Choose one of the 100 patients at random, each equally likely. Let C be the event “the patient has the condition” and T the event “the patient tests positive.” The table gives

$$P(C) = \frac{20}{100}, \quad P(T) = \frac{26}{100}, \quad P(C \cap T) = \frac{18}{100}.$$

Conditional Probability

If a patient has the condition, how likely is the test to catch it? This is $P(T \mid C)$: Knowing that C occurs shrinks the group to the first row: 20 patients, 18 of whom test positive. So

$$P(T \mid C) = \frac{|T \cap C|}{|C|} = \frac{18}{20} = 90\%.$$

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If a patient tests positive, how likely is it that they have the condition? This is $P(C \mid T)$: Now the group is the first column: 26 patients, 18 of whom have the condition. So

$$P(C \mid T) = \frac{|C \cap T|}{|T|} = \frac{18}{26} \approx 69\%.$$

Conditional Probability

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$$P(C \mid T) = \frac{|C \cap T|}{|T|} = \frac{18}{26} \approx 69\%.$$

Same intersection, different denominators, very different answers. Mixing up $P(T \mid C)$ and $P(C \mid T)$ is a common, serious mistake: the test's maker reports the first number, but a patient who tests positive needs the second.

Check Your Understanding

Using the screening-test table, find each of the following.

- (a) $P(T^c \mid C^c)$: if a patient does not have the condition, how likely is the test to come back negative?
- (b) $P(C \mid T^c)$: if a patient tests negative, how likely is it that they have the condition anyway?

Check Your Understanding

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- (a) $P(T^c \mid C^c)$: if a patient does not have the condition, how likely is the test to come back negative?
- (b) $P(C \mid T^c)$: if a patient tests negative, how likely is it that they have the condition anyway?

Solution:

- (a) Knowing C^c shrinks the group to the second row: 80 patients without the condition, 72 of whom test negative. So $P(T^c \mid C^c) = \frac{72}{80} = 0.90 = 90\%$.
- (b) Knowing T^c shrinks the group to the second column: 74 patients who tested negative, 2 of whom have the condition. So $P(C \mid T^c) = \frac{2}{74} \approx 0.027 = 2.7\%$.

A negative result is reassuring: the probability that the patient has the condition drops from 20% before the test to about 2.7%.

Conditional Probability

Example: A certain condition affects 1 in every 1,000 people. A screening test for the condition is 99% accurate in both directions: if a patient has the condition, the test comes back positive 99% of the time, and if a patient does not have the condition, the test comes back negative 99% of the time.

A patient's screening test comes back positive. What is the probability that the patient has the condition?

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A patient's screening test comes back positive. What is the probability that the patient has the condition?

Most people, including many trained clinicians, answer “about 99%.” Make your own guess before we work it out.

Conditional Probability

Solution: We build a two-way table for a group of 100,000 patients. Since 1 in 1,000 people has the condition, 100 of the patients have it and the other 99,900 do not. Of the 100 patients with the condition, 99%, or 99 patients, test positive, and 1 tests negative. Of the 99,900 patients without the condition, 99%, or 98,901 patients, test negative, and the other 1%, or 999 patients, test positive.

Conditional Probability

Solution: We build a two-way table for a group of 100,000 patients. Since 1 in 1,000 people has the condition, 100 of the patients have it and the other 99,900 do not. Of the 100 patients with the condition, 99%, or 99 patients, test positive, and 1 tests negative. Of the 99,900 patients without the condition, 99%, or 98,901 patients, test negative, and the other 1%, or 999 patients, test positive.

	Test positive	Test negative	Total
Condition present	99	1	100
Condition absent	999	98,901	99,900
Total	1,098	98,902	100,000

Conditional Probability

Let C be the event “the patient has the condition” and T the event “the patient tests positive.” Knowing T shrinks the group to the first column: 1,098 patients, only 99 of whom have the condition. So

$$P(C \mid T) = \frac{99}{1,098} \approx 0.09 = 9\%.$$

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Fewer than 1 in 10 positive results is real. The reason is that the condition is so rare: even a 1% error rate, applied to the 99,900 healthy patients, produces 999 false positives, many more than the 99 true positives.

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Fewer than 1 in 10 positive results is real. The reason is that the condition is so rare: even a 1% error rate, applied to the 99,900 healthy patients, produces 999 false positives, many more than the 99 true positives.

The test is still useful. Before the test, the patient’s probability of having the condition was $\frac{1}{1,000} = 0.1\%$, and the positive result raises it to 9%, about 90 times as high. A positive screening result is a reason for follow-up testing, not a diagnosis.

Conditional Probability

Example: The blood type probabilities fit in a two-way table, with letter types for the rows, the Rh factor for the columns, and probabilities instead of counts in the cells.

	Rh positive	Rh negative	Total
Type A	0.32	0.08	0.40
Type B	0.08	0.02	0.10
Type AB	0.04	0.01	0.05
Type O	0.36	0.09	0.45
Total	0.80	0.20	1

If we know that the person has type A blood, what is the probability that they are Rh negative?

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Type O	0.36	0.09	0.45
Total	0.80	0.20	1

If we know that the person has type A blood, what is the probability that they are Rh negative?

Solution: Let N be the event “Rh negative” and Z the event “type A.” Knowing Z shrinks the sample space to the first row, where $N \cap Z = \{A-\}$, so $P(N | Z) = \frac{P(A-)}{P(Z)} = \frac{0.08}{0.40} = 0.20$.

Conditional Probability

Notice that $P(N \mid Z) = 0.20 = P(N)$. The same is true in every row:

$$\frac{0.08}{0.40} = \frac{0.02}{0.10} = \frac{0.01}{0.05} = \frac{0.09}{0.45} = 0.20.$$

Knowing a person's letter type does not change the probability that they are Rh negative, so the letter type and the Rh factor are independent. In the table, each cell is the product of its row total and its column total: for example, $P(O+) = 0.45 \cdot 0.80 = 0.36$.

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Compare the screening-test table, where knowing whether a patient has the condition changes the probability of a positive test from 26% to 90%. A useful test must be very far from independent of the condition it tests for.

Conditional Probability

Multiplying both sides of the definition of conditional probability by $P(B)$ gives a way to find $P(A \cap B)$ even when A and B are not independent. Since $A \cap B = B \cap A$, we can also condition on A instead.

The General Multiplication Rule

If A and B are events with $P(A) \neq 0$ and $P(B) \neq 0$, then

$$P(A \cap B) = P(B) \cdot P(A \mid B) = P(A) \cdot P(B \mid A).$$

This is called **the General Multiplication Rule**. In words: for both events to occur, B must occur, and then A must occur in the smaller world where B has occurred.

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This is called **the General Multiplication Rule**. In words: for both events to occur, B must occur, and then A must occur in the smaller world where B has occurred.

If A and B are independent, then $P(A \mid B) = P(A)$, and the General Multiplication Rule becomes the Multiplication Rule.

Conditional Probability

The General Multiplication Rule is a “fraction of a fraction” calculation.

Example: At a clinic, 30% of the patients have diabetes, and 40% of the patients with diabetes also have high blood pressure. What fraction of all of the clinic’s patients have both?

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Example: At a clinic, 30% of the patients have diabetes, and 40% of the patients with diabetes also have high blood pressure. What fraction of all of the clinic’s patients have both?

Solution: Let D be the event “the patient has diabetes” and H the event “the patient has high blood pressure.” We are told that $P(D) = 0.30$ and $P(H \mid D) = 0.40$. By the General Multiplication Rule,

$$P(H \cap D) = P(D) \cdot P(H \mid D) = 0.30 \cdot 0.40 = 0.12,$$

so 12% of the clinic’s patients have both.

Conditional Probability

Example: A bag contains 2 red marbles, 2 yellow marbles, and 2 blue marbles. Two marbles are drawn without putting the first one back. What is the probability that both are red?

Conditional Probability

Example: A bag contains 2 red marbles, 2 yellow marbles, and 2 blue marbles. Two marbles are drawn without putting the first one back. What is the probability that both are red?

Solution: Let R_1 be the event “the first marble is red” and R_2 the event “the second marble is red.” Two of the 6 marbles are red, so $P(R_1) = \frac{2}{6} = \frac{1}{3}$. If the first marble is red, then 1 red marble is left among the 5 marbles in the bag, so $P(R_2 \mid R_1) = \frac{1}{5}$. By the General Multiplication Rule,

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 \mid R_1) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}.$$

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$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 \mid R_1) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}.$$

Compare this with drawing with replacement, where the draws are independent and $P(R_1 \cap R_2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Check Your Understanding

At a blood drive, 9 of the 20 donors have type O blood. Two of the donors are chosen at random, one after the other, to be interviewed for a newsletter. What is the probability that both have type O blood?

Check Your Understanding

At a blood drive, 9 of the 20 donors have type O blood. Two of the donors are chosen at random, one after the other, to be interviewed for a newsletter. What is the probability that both have type O blood?

Solution: Let O_1 be the event “the first donor chosen has type O blood” and O_2 the event “the second donor chosen has type O blood.” Then $P(O_1) = \frac{9}{20}$. If the first donor chosen has type O blood, then 8 of the remaining 19 donors do, so $P(O_2 \mid O_1) = \frac{8}{19}$. By the General Multiplication Rule,

$$P(O_1 \cap O_2) = P(O_1) \cdot P(O_2 \mid O_1) = \frac{9}{20} \cdot \frac{8}{19} = \frac{72}{380} = \frac{18}{95} \approx 0.189.$$

For two unrelated residents of the city, whose blood types are independent, the probability would be $0.45 \cdot 0.45 = 0.2025$. Here the second choice depends on the first, since a donor cannot be chosen twice.

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Solution: The event is $\{RY, YR\}$. The draws are not independent, so we use the General Multiplication Rule for each order:

$$P(RY) = \frac{4}{16} \cdot \frac{7}{15} = \frac{7}{60} \quad \text{and} \quad P(YR) = \frac{7}{16} \cdot \frac{4}{15} = \frac{7}{60}.$$

For example, in RY, 4 of the 16 marbles are red, and then 7 of the remaining 15 are yellow. The two orders are mutually exclusive, so by the Addition Rule, the probability is

$$\frac{7}{60} + \frac{7}{60} = \frac{7}{30} \approx 0.233.$$

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A bag contains 4 red marbles, 7 yellow marbles, and 5 blue marbles. Three marbles are drawn, and no marble is put back. What is the probability of getting 2 red marbles and 1 yellow marble?

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Solution: The event is $\{RRY, RYR, YRR\}$. Using the General Multiplication Rule twice, each draw is conditioned on everything drawn before it. For RRY: 4 of the 16 marbles are red, then 3 of the remaining 15 are red, then 7 of the remaining 14 are yellow. So

$$P(RRY) = \frac{4}{16} \cdot \frac{3}{15} \cdot \frac{7}{14} = \frac{1}{40}, \quad P(RYR) = \frac{4}{16} \cdot \frac{7}{15} \cdot \frac{3}{14} = \frac{1}{40}, \quad P(YRR) = \frac{7}{16} \cdot \frac{4}{15} \cdot \frac{3}{14} = \frac{1}{40}.$$

The three orders are mutually exclusive, so by the Addition Rule, the probability is

$$\frac{1}{40} + \frac{1}{40} + \frac{1}{40} = \frac{3}{40} = 0.075.$$

Binomial Experiments

A **binomial experiment** is an experiment made up of a fixed number n of repeated **trials**, in which each trial has exactly two possible results, called *success* and *failure*, the trials are independent, and the probability of success, written p , is the same on every trial. In a binomial experiment, we are interested in the number of successes.

Binomial Experiments

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Example: Toss a fair coin 10 times and count the heads. Here $n = 10$, a success is heads, and $p = \frac{1}{2}$.

Example: A treatment works for 80% of the patients who receive it. Give it to 4 unrelated patients and count how many respond. Here $n = 4$, a success is a patient who responds, and $p = 0.8$.

Example: Draw 3 marbles from the bag of 4 red, 7 yellow, and 5 blue marbles, putting each one back, and count the red marbles. Here $n = 3$, a success is a red marble, and $p = \frac{4}{16} = \frac{1}{4}$.

Binomial Experiments

Example: A fair coin is tossed 4 times. What is the probability of getting exactly 2 heads?

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Solution: By the Multiplication Rule, each sequence of 4 tosses has probability $(\frac{1}{2})^4 = \frac{1}{16}$.

A sequence with exactly 2 heads is determined by choosing which 2 of the 4 tosses are heads, so there are ${}_4C_2 = 6$ such sequences:

HHTT, HTHT, HTTH, THHT, THTH, TTHH.

Let E_2 be the event “exactly 2 heads,” which consists of these six sequences. They are mutually exclusive, so by the Addition Rule,

$$P(E_2) = {}_4C_2 \cdot \frac{1}{16} = \frac{6}{16} = \frac{3}{8}.$$

Binomial Experiments

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Solution: Write R for a patient who responds and N for a patient who does not. By the Multiplication Rule, the sequence RRRN has probability

$$0.8 \cdot 0.8 \cdot 0.8 \cdot 0.2 = (0.8)^3(0.2) = 0.1024.$$

Every sequence with 3 R's and 1 N has this same probability, since it has the same factors in a different order. Let E_3 be the event “exactly 3 patients respond.” There are ${}_4C_3 = 4$ sequences in E_3 , one for each choice of which 3 patients respond, and they are mutually exclusive. By the Addition Rule,

$$P(E_3) = {}_4C_3 \cdot (0.8)^3(0.2) = 4 \cdot 0.1024 = 0.4096.$$

Binomial Experiments

The same reasoning works for any binomial experiment. Each sequence of n trials with exactly k successes has k factors of p and $n - k$ factors of $1 - p$, so by the Multiplication Rule it has probability $p^k(1 - p)^{n-k}$. There are ${}_nC_k$ such sequences, one for each choice of which k trials are successes, and they are mutually exclusive.

Binomial Probabilities

In a binomial experiment with n trials and probability of success p , let E_k be the event “exactly k successes.” Then

$$P(E_k) = {}_nC_k p^k (1 - p)^{n-k}.$$

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Binomial Probabilities

In a binomial experiment with n trials and probability of success p , let E_k be the event “exactly k successes.” Then

$$P(E_k) = {}_nC_k p^k(1 - p)^{n-k}.$$

For a fair coin, $p = 1 - p = \frac{1}{2}$, so $p^k(1 - p)^{n-k} = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$. If E_k is the event “exactly k heads in n tosses,” then

$$P(E_k) = \frac{{}_nC_k}{2^n}.$$

Binomial Experiments

Example: For 4 tosses of a fair coin, let E_k be the event “exactly k heads.” Then

Number of heads k	0	1	2	3	4
$P(E_k)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

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Every sequence of tosses is in exactly one of the events E_0, E_1, E_2, E_3, E_4 , so their probabilities add up to 1:

$$\sum_{k=0}^4 P(E_k) = \sum_{k=0}^4 \frac{{}^4C_k}{16} = \frac{1 + 4 + 6 + 4 + 1}{16} = \frac{16}{16} = 1.$$

Binomial Experiments

Example: A fair coin is tossed 10 times. What is the probability of getting exactly 5 heads?
At least 9 heads? At most 8 heads?

Binomial Experiments

Example: A fair coin is tossed 10 times. What is the probability of getting exactly 5 heads? At least 9 heads? At most 8 heads?

Solution: Let E_k be the event “exactly k heads.” There are $2^{10} = 1,024$ sequences, so $P(E_5) = \frac{{}_{10}C_5}{1,024} = \frac{252}{1,024} \approx 0.246$. The event “at least 9 heads” is $E_9 \cup E_{10}$, so by the Addition Rule,

$$P(E_9 \cup E_{10}) = P(E_9) + P(E_{10}) = \frac{{}_{10}C_9 + {}_{10}C_{10}}{1,024} = \frac{11}{1,024} \approx 0.011.$$

The event “at most 8 heads” is the complement of $E_9 \cup E_{10}$, so

$$P((E_9 \cup E_{10})^c) = 1 - \frac{11}{1,024} \approx 0.989.$$

Even the most likely number of heads, 5, happens less than a quarter of the time.

Check Your Understanding

The weighted coin from before lands on heads with probability $\frac{2}{3}$. It is tossed 3 times. What is the probability of getting exactly 2 heads?

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The weighted coin from before lands on heads with probability $\frac{2}{3}$. It is tossed 3 times. What is the probability of getting exactly 2 heads?

Solution: This is a binomial experiment with $n = 3$ and $p = \frac{2}{3}$, so $1 - p = \frac{1}{3}$. The probability of exactly 2 heads is

$${}_3C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^1 = 3 \cdot \frac{4}{9} \cdot \frac{1}{3} = \frac{12}{27} = \frac{4}{9}.$$

Check Your Understanding

A vaccine produces immunity in 90% of the patients who receive it. Five unrelated patients are vaccinated.

- (a) What is the probability that all 5 become immune?
- (b) What is the probability that exactly 4 become immune?
- (c) What is the probability that at least 4 become immune?

Check Your Understanding

A vaccine produces immunity in 90% of the patients who receive it. Five unrelated patients are vaccinated.

- (a) What is the probability that all 5 become immune?
- (b) What is the probability that exactly 4 become immune?
- (c) What is the probability that at least 4 become immune?

Solution: This is a binomial experiment with $n = 5$ and $p = 0.9$.

- (a) ${}_5C_5(0.9)^5(0.1)^0 = (0.9)^5 \approx 0.590$.
- (b) ${}_5C_4(0.9)^4(0.1)^1 = 5 \cdot 0.6561 \cdot 0.1 \approx 0.328$.
- (c) The events in parts (a) and (b) are mutually exclusive, so by the Addition Rule, the probability is about $0.590 + 0.328 = 0.918$.

Check Your Understanding

A fair coin is tossed 10 times. What is the probability of getting at least 8 heads?

Check Your Understanding

A fair coin is tossed 10 times. What is the probability of getting at least 8 heads?

Solution: Let E_k be the event “exactly k heads.” The event “at least 8 heads” is $E_8 \cup E_9 \cup E_{10}$, and these three events are mutually exclusive. By the Addition Rule,

$$P(E_8 \cup E_9 \cup E_{10}) = \frac{{}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}}{1,024} = \frac{45 + 10 + 1}{1,024} = \frac{56}{1,024} \approx 0.055.$$

Check Your Understanding

In the city from the blood type experiment, 45% of the residents have type O blood and 9% have type O—.

- (a) Five unrelated residents donate blood. What is the probability that exactly 2 of them have type O blood?
- (b) Ten unrelated residents donate blood. What is the probability that at least one of them has type O— blood?

Check Your Understanding

In the city from the blood type experiment, 45% of the residents have type O blood and 9% have type O—.

- (a) Five unrelated residents donate blood. What is the probability that exactly 2 of them have type O blood?
- (b) Ten unrelated residents donate blood. What is the probability that at least one of them has type O— blood?

Solution:

- (a) This is a binomial experiment with $n = 5$ and $p = 0.45$, so the probability is

$${}_5C_2 (0.45)^2 (0.55)^3 = 10 \cdot 0.2025 \cdot 0.166375 \approx 0.337.$$

- (b) This is a binomial experiment with $n = 10$ and $p = 0.09$. The probability that none of the ten has type O— blood is ${}_{10}C_0 (0.09)^0 (0.91)^{10} = (0.91)^{10} \approx 0.389$, so by complements, the probability that at least one does is about $1 - 0.389 = 0.611$.

Probability

Tool	Use it when	Formula
Probability by Counting	the sample points are equally likely	$P(A) = \frac{ A }{ S }$
The Addition Rule	the events are mutually exclusive	$P(A \cup B) = P(A) + P(B)$
Complements	the complement is easier to work with	$P(A^c) = 1 - P(A)$
The Inclusion-Exclusion Principle for Events	the events overlap	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
The Multiplication Rule	the events are independent	$P(A \cap B) = P(A) \cdot P(B)$
Conditional Probability	we know that B occurs	$P(A B) = \frac{P(A \cap B)}{P(B)}$
The General Multiplication Rule	the events may not be independent	$P(A \cap B) = P(B) \cdot P(A B)$
Binomial Probabilities	counting successes in independent trials	${}_nC_k p^k (1 - p)^{n-k}$