

Counting

MATH 110: General Education Math

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Counting

- 1 Sets
- 2 The Fundamental Principle of Counting
- 3 Factorials
- 4 Permutations
- 5 Combinations
- 6 Counting Arguments

A **set** is a collection of distinct objects, called its **elements**. We describe a set by listing its elements between braces, and we usually name sets with capital letters. For example, the possible results of rolling a die form the set

$$D = \{1, 2, 3, 4, 5, 6\}.$$

We write $x \in S$, read “ x in S ” to denote that x is an element of a set S . We write $x \notin S$, read “ x not in S ,” to denote that x is not an element of S . For example, for the set D above, we have $3 \in D$ and $7 \notin D$.

Two rules for listing the elements of a set:

- The order in which the elements are listed does not matter: $\{1, 2, 3\} = \{3, 1, 2\}$.
- Each element is listed only once: we never write $\{1, 1, 2\}$ (but if we did by accident, it would mean $\{1, 2\}$).

The number of elements in a set A is called its **cardinality**, written $|A|$. A set with n elements is called an **n -element set**.

Example: Five nurses – Abby, Bianca, Claudia, Dulce, and Elisa – are on shift. Using their first initials, the set of nurses on shift can be represented by the set

$$N = \{A, B, C, D, E\},$$

so $|N| = 5$, and N is a 5-element set.

The **empty set**, written $\emptyset$, is the set with no elements at all, so $|\emptyset| = 0$. For example, the set of nurses on the overnight shift at a clinic that is closed overnight is $\emptyset$.

Check Your Understanding

Write each set by listing its elements, and find its cardinality.

- (a) C , the possible results of tossing a coin twice.
- (b) W , the days of the week a clinic is open, if it is open Monday through Friday.
- (c) E , the results of rolling a die that are greater than 6.

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- (a) C , the possible results of tossing a coin twice.
- (b) W , the days of the week a clinic is open, if it is open Monday through Friday.
- (c) E , the results of rolling a die that are greater than 6.

Solution:

- (a) $C = \{HH, HT, TH, TT\}$, so $|C| = 4$.
- (b) $W = \{\text{Mon, Tue, Wed, Thu, Fri}\}$, so $|W| = 5$.
- (c) No result of rolling a die is greater than 6, so $E = \emptyset$ and $|E| = 0$.

A set A is a **subset** of a set B , written $A \subseteq B$, if every element of A is also an element of B .

Example: Three of the five nurses in $N = \{A, B, C, D, E\}$ are sent to a seminar. The seminar team

$$T = \{B, C, E\}$$

is a subset of N : $T \subseteq N$.

Two facts we will use later:

- Every set is a subset of itself: $N \subseteq N$.
- The empty set is a subset of every set: $\emptyset \subseteq N$. It has no elements, so none of its elements can fail to be in N .

Check Your Understanding

Three nurses – Abby, Bianca, and Claudia – form the set $S = \{A, B, C\}$. List every subset of S . How many are there?

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Solution: Sorting the subsets by their number of elements:

Number of elements	Subsets	How many
0	$\emptyset$	1
1	$\{A\}, \{B\}, \{C\}$	3
2	$\{A, B\}, \{A, C\}, \{B, C\}$	3
3	$\{A, B, C\}$	1

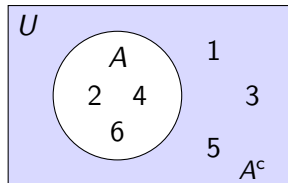
There are $1 + 3 + 3 + 1 = 8$ subsets in all.

In a given problem, all of the sets we work with are usually subsets of one larger set, called the **universal set** and written U .

The **complement** of a set A , written A^c , is the set of all elements of U that are not in A .

Example: Let $U = \{1, 2, 3, 4, 5, 6\}$ be the possible results of rolling a die, and let $A = \{2, 4, 6\}$ be the even results. Then

$$A^c = \{1, 3, 5\}.$$



A picture like this, with the universal set drawn as a rectangle and its subsets drawn as circles, is called a **Venn diagram**.

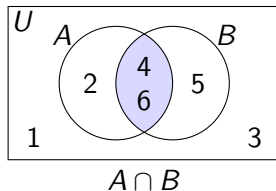
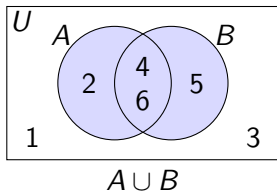
Sets

The **union** of sets A and B , written $A \cup B$, is the set of elements that are in A or in B (or in both).

The **intersection** of A and B , written $A \cap B$, is the set of elements that are in both A and B .

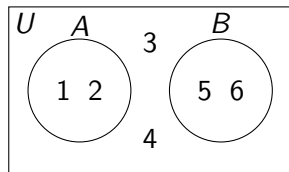
Example: Roll a die, so $U = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{2, 4, 6\}$ (even) and $B = \{4, 5, 6\}$ (greater than 3). Then

$$A \cup B = \{2, 4, 5, 6\} \quad \text{and} \quad A \cap B = \{4, 6\}.$$



Sets A and B are **disjoint** if they have no elements in common, that is, if $A \cap B = \emptyset$.

Example: Roll a die, and let $A = \{1, 2\}$ and $B = \{5, 6\}$. Then $A \cap B = \emptyset$, so A and B are disjoint.



Every set is disjoint from its complement, and together they make up the universal set: $A \cap A^c = \emptyset$ and $A \cup A^c = U$.

Check Your Understanding

A ward has six patients: Ana, Ben, Carla, David, Eva, and Frank. Using first initials, $U = \{A, B, C, D, E, F\}$. The patients taking a blood pressure medication form the set $H = \{A, C, D\}$, and the patients over 65 form the set $S = \{C, D, F\}$.

- (a) Find $H \cup S$, $H \cap S$, and H^c .
- (b) Are H and S disjoint?
- (c) Compare $|H \cup S|$ with $|H| + |S|$. Why are they different?

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- (a) Find $H \cup S$, $H \cap S$, and H^c .
- (b) Are H and S disjoint?
- (c) Compare $|H \cup S|$ with $|H| + |S|$. Why are they different?

Solution:

- (a) $H \cup S = \{A, C, D, F\}$, $H \cap S = \{C, D\}$, and $H^c = \{B, E, F\}$.
- (b) No: $H \cap S = \{C, D\} \neq \emptyset$.
- (c) $|H \cup S| = 4$, but $|H| + |S| = 3 + 3 = 6$. Carla and David are in both sets, so adding $|H|$ and $|S|$ counts each of them twice.

The Fundamental Principle of Counting

Example: You toss a coin three times. What are the possibilities?

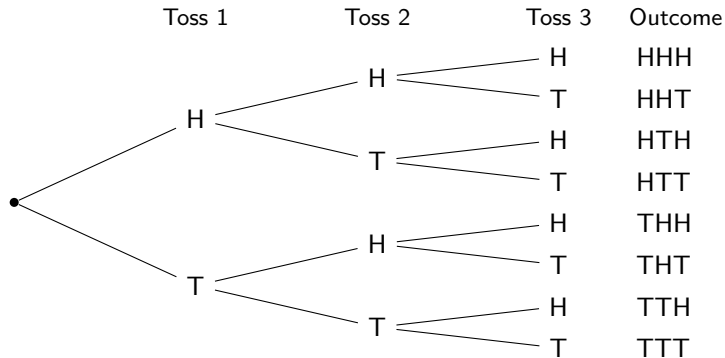
Solution: We can count each possibility explicitly. We could get three heads (HHH), two heads followed by a tail (HHT), a head followed by a tail followed by a head (HTH), etc.

All in all, we get 8 possibilities:

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

The Fundamental Principle of Counting

A useful way to visualize the solution of the coin-tossing example is via a **tree diagram**.



Each toss has two possible outcomes. Note that the total number of possibilities is $2 \cdot 2 \cdot 2 = 8$.

The Fundamental Principle of Counting

Example: A hospital cafeteria offers your choice of one item from each category.

Entrée

Chicken, Pasta, Salad

Side

Soup, Fruit

Drink

Water, Juice

How many different three-item meals are possible?

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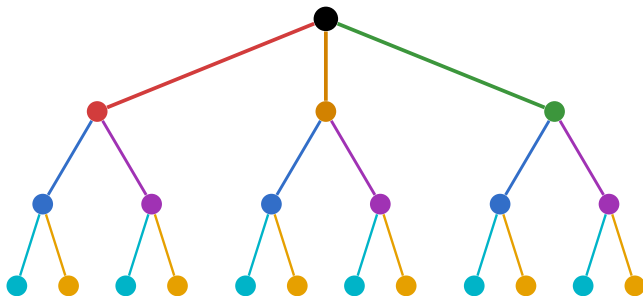
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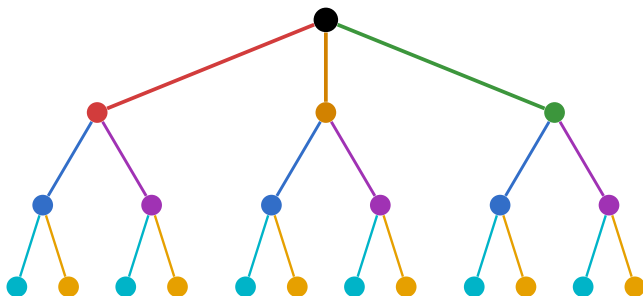
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Soup, Fruit

Drink

Water, Juice

How many different three-item meals are possible?



Solution: There are $3 \cdot 2 \cdot 2 = 12$ possible meals.

The Fundamental Principle of Counting

The previous examples hint at a more general result, called **the Fundamental Principle of Counting**.

The Fundamental Principle of Counting

If a first task can be completed in m ways, and for each way of completing the first task a second task can be completed in n ways, then the total number of ways to complete both tasks in order is $m \cdot n$.

More generally, if k tasks are completed in order, and the tasks can be completed in $m_1, m_2, m_3, \dots, m_k$ ways, respectively, no matter how the earlier tasks are completed, then the total number of ways to complete all k tasks in order is

$$m_1 \cdot m_2 \cdot m_3 \cdot \dots \cdot m_k.$$

Check Your Understanding

A clinic's patient intake form asks each patient to choose one insurance type (Private, Medicaid, Medicare, or Uninsured), one preferred language (English or Spanish), and one appointment time (Morning, Afternoon, or Evening). In how many different ways can the form be filled out?

Check Your Understanding

A clinic's patient intake form asks each patient to choose one insurance type (Private, Medicaid, Medicare, or Uninsured), one preferred language (English or Spanish), and one appointment time (Morning, Afternoon, or Evening). In how many different ways can the form be filled out?

Solution: Filling out the form consists of three tasks: choosing an insurance type (4 ways), choosing a language (2 ways), and choosing an appointment time (3 ways). By the Fundamental Principle of Counting, the form can be filled out in

$$4 \cdot 2 \cdot 3 = 24$$

different ways.

Check Your Understanding

A coin is tossed 10 times, and the result of each toss is recorded in order (for example, HTTHHHTHTT). How many different sequences of heads and tails are possible?

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Solution: Each of the 10 tosses is a task that can be completed in 2 ways. By the Fundamental Principle of Counting, the number of possible sequences is

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{10 \text{ factors}} = 2^{10} = 1,024.$$

A tree diagram for this experiment would end in 1,024 branches!

Check Your Understanding

A coin is tossed n times, where n is any positive whole number, and the result of each toss is recorded in order. How many different sequences of heads and tails are possible?

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$$\underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{n \text{ factors}} = 2^n.$$

This agrees with what we found before: $2^3 = 8$ sequences for 3 tosses, and $2^{10} = 1,024$ sequences for 10 tosses.

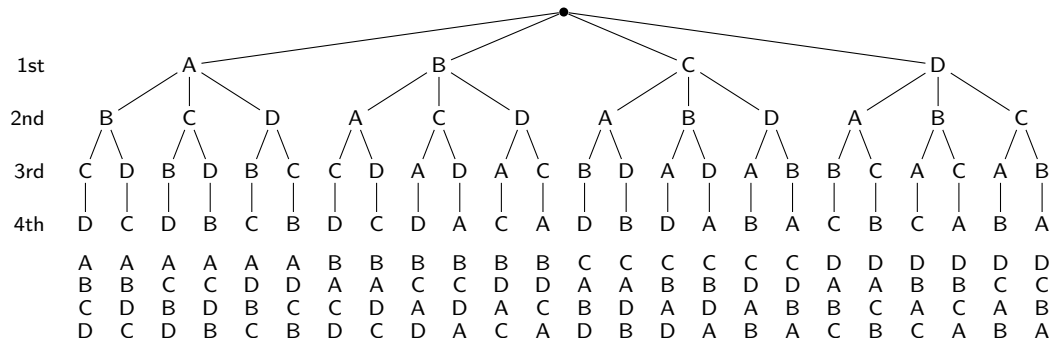
Factorials

Example: You have 4 different trophies to be displayed in a row on a shelf. How many different arrangements are possible?

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Solution: Label the trophies A, B, C, and D. The tree diagram is shown below.



Total: 24

Factorials

There is something different about the previous example than the ones before it: the possibilities are different at different nodes on the same level. For example, if the first trophy is B then the second trophy can be A, C, or D, whereas if the first trophy is C then the second trophy can be A, B, or D. Crucially, though, the *number* of possibilities at each node on the same level *is* equal.

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There are 4 choices for the first trophy. For each choice of the first trophy there are 3 choices for the second trophy. For each choice of first and second trophies, there are 2 choices for the third trophy. For each choice of the first, second, and third trophies there is 1 choice for the fourth trophy – it is forced. Thus, the Fundamental Principle of Counting applies, and the total number of arrangements is

$$4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

Factorials

Products like the one in the previous example ($4 \cdot 3 \cdot 2 \cdot 1$) occur so frequently in counting problems that they get a special name: **factorials**. By convention, $0!$ – read “zero factorial” – is defined to be 1 (we will see why this makes sense later). Then, higher factorials are calculated as follows.

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \cdot 0! &= 1 \cdot 1 &= 1 \\ 2! &= 2 \cdot 1! &= 2 \cdot 1 &= 2 \\ 3! &= 3 \cdot 2! &= 3 \cdot 2 \cdot 1 &= 6 \\ 4! &= 4 \cdot 3! &= 4 \cdot 3 \cdot 2 \cdot 1 &= 24 \\ &\vdots \\ n! &= n(n-1)! &= n(n-1)(n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

In words, $n!$ – read “ n factorial” – is the product of the first n positive integers.

Note: factorials grow very quickly! Even just $10!$ is equal to 3,628,800.

Permutations

Each of the possibilities in the trophy-arranging example (ABCD, ABDC, ACBD, etc.) is called a *permutation* of the set $\{A, B, C, D\}$.

In general, a **permutation** of a set of elements is an arrangement of those elements in which *order matters*.

There are $n!$ ways to permute n elements.

This follows directly from the Fundamental Principle of Counting: there are n choices for the first element, then $n - 1$ choices for the next element, then $n - 2$ choices for the next element, and so on, and so the total number of permutations is

$$n(n - 1)(n - 2) \cdot \dots \cdot 3 \cdot 2 \cdot 1 = n!$$

Check Your Understanding

A nurse has 5 patients to check on during her morning rounds, and she sees them one at a time. In how many different orders can she see her 5 patients?

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A nurse has 5 patients to check on during her morning rounds, and she sees them one at a time. In how many different orders can she see her 5 patients?

Solution: Each order is a permutation of the 5 patients. There are 5 choices for the first patient, then 4 choices for the second, 3 for the third, 2 for the fourth, and 1 for the last. By the Fundamental Principle of Counting, the number of orders is

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

If she used a different order every morning, it would take her 120 days – about four months – to try them all.

Permutations

Example: You have 4 trophies to display in a row on a shelf, but there is only enough room for two of them. How many different arrangements are possible?

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Solution: There are 4 choices for the first trophy. For each choice of the first trophy there are 3 choices for the second trophy. Thus, by the Fundamental Principle of Counting, there are

$$4 \cdot 3 = 12$$

ways to display two of the four trophies.

For reasons we will see shortly, it can be useful to write this as

$$4 \cdot 3 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{4!}{2!} = \frac{4!}{(4-2)!}.$$

Permutations

For integers n and k with $0 \leq k \leq n$, a k -**permutation** of a set of n elements is an arrangement of k of the n elements in which *order matters*.

For example, in the previous example we counted the number of 2-permutations of a 4-element set.

A natural question to ask is: how many k -permutations of n elements are there? Ideally, we'd like a formula that involves only n and k .

Permutations

For the answer, we turn again to the Fundamental Principle of Counting.

There are n choices for the first element. For each choice of the first element there are then $n - 1$ choices for the second element. For each choice of the first and second elements, there are $n - 2$ choices for the third element, and so on. Therefore the number of k -permutations of n elements is

$$n(n-1)(n-2) \cdot \dots \cdot (n-k+2)(n-k+1).$$

This is called a **permutation number** and is denoted ${}_nP_k$, read “ n permute k .” We can do a bit of algebra to write it in a compact form involving only n and k :

$${}_nP_k = \frac{n(n-1) \cdot \dots \cdot (n-k+2)(n-k+1)(n-k)(n-k-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1}{(n-k)(n-k-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1} = \frac{n!}{(n-k)!}.$$

Permutation Numbers

The permutation number ${}_nP_k$, which gives the number of k -permutations of n elements ($0 \leq k \leq n$), is given by

$${}_nP_k = \frac{n!}{(n-k)!}.$$

Example: Four shifts need to be assigned out of a pool of 10 nurses. Each nurse can only take one of the four shifts. How many different schedules are there?

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Solution: Each schedule is a 4-permutation of the 10 nurses in the pool. Therefore, the number of different schedules is

$${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5,040.$$

Check Your Understanding

A clinic has 5 medical assistants working today. One is needed to staff the front desk, one to take vital signs, and one to draw blood, and no assistant can do more than one of these jobs. In how many different ways can the three jobs be filled?

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Solution: The jobs are different, so order matters: each way of filling them is a 3-permutation of the 5 assistants. Therefore, the number of ways is

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60.$$

Check Your Understanding

Each nurse in a hospital ward is assigned a four-letter access code to a medicine cabinet. Letters in an access code can't be repeated. How many possible access codes are there?

Check Your Understanding

Each nurse in a hospital ward is assigned a four-letter access code to a medicine cabinet. Letters in an access code can't be repeated. How many possible access codes are there?

Solution: Each access code is a 4-permutation of the 26 letters of the alphabet. Therefore, there are

$${}_{26}P_4 = \frac{26!}{(26-4)!} = \frac{26!}{22!} = 26 \cdot 25 \cdot 24 \cdot 23 = 358,800$$

possible access codes.

Check Your Understanding

You have 6 trophies.

- (a) The shelf has room for all 6 of them in a row. How many different arrangements are possible? Compute ${}_6P_6$ and compare.
- (b) The shelf is being repainted, so there is no room for any trophies at all. In how many ways can 0 of the 6 trophies be displayed? Compute ${}_6P_0$ and compare.

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Solution:

- (a) There are $6! = 720$ arrangements of all 6 trophies. The formula agrees:
$${}_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{720}{1} = 720.$$
- (b) There is exactly one way: leave the shelf empty. The formula agrees again:
$${}_6P_0 = \frac{6!}{(6-0)!} = \frac{6!}{6!} = 1.$$

Part (a) only works because $0! = 1$ – this is one reason for that convention.

Example: Three nurses out of a pool of 5 – Abby, Bianca, Claudia, Dulce, and Elisa – are to be sent to a seminar. One will be the team leader, one will take notes, and one will present. How many different teams are there?

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Solution: The three roles are different, so order matters. For example, listing the team leader first, the note-taker second, and the presenter third, the teams BCE and CBE are different: in the first, Bianca leads the team, and in the second, Claudia does. Each team is a 3-permutation of the 5 nurses, so the number of different teams is

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60.$$

Combinations

Example: Again, three of the 5 nurses – Abby, Bianca, Claudia, Dulce, and Elisa – are to be sent to a seminar, but this time with no distinct roles. How many different teams are there?

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For example, the team consisting of Bianca, Claudia, and Elisa is counted six times, once for each way of assigning the three roles: BCE, BEC, CBE, CEB, EBC, and ECB.

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For example, the team consisting of Bianca, Claudia, and Elisa is counted six times, once for each way of assigning the three roles: BCE, BEC, CBE, CEB, EBC, and ECB.

In fact, *every* three-person team is counted $3! = 6$ times in ${}_5P_3$. So, to get the total number of *different* teams, we must divide by $3!$:

$$\frac{{}_5P_3}{3!} = \frac{60}{6} = 10.$$

There are 10 different teams.

Combinations

A **combination** is a choice of elements of a set in which order does not matter.

If n and k are integers with $0 \leq k \leq n$, then a k -**combination** of n elements is a choice of k of the n elements in which order does not matter.

The number of k -combinations of n elements is called a **combination number** and denoted ${}_nC_k$, read “ n choose k .” Since in the permutation number ${}_nP_k$ each k -combination is counted exactly $k!$ times, we have

$${}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Combination Numbers

The combination number ${}_nC_k$, which gives the number of k -combinations of n elements ($0 \leq k \leq n$), is given by

$${}_nC_k = \frac{n!}{k!(n-k)!}.$$

Note: combination numbers are also sometimes referred to as *binomial coefficients*.

Example: A clinic will send 4 of its 9 medical assistants to a CPR recertification class. The assistants who go have no distinct roles. How many different groups of 4 can the clinic send?

Combinations

Example: A clinic will send 4 of its 9 medical assistants to a CPR recertification class. The assistants who go have no distinct roles. How many different groups of 4 can the clinic send?

Solution: Order does not matter, so each group is a 4-combination of the 9 assistants. Therefore, the number of different groups is

$${}_9C_4 = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{4! \cdot 5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{3,024}{24} = 126.$$

Notice that the $5!$ cancels, so there is no need to compute $9!$.

Check Your Understanding

A nurse needs to make follow-up calls to 2 of her 6 patients today. How many different pairs of patients can she choose? Check your answer by listing the pairs, using the patients' initials A, B, C, D, E, and F.

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Solution: Order does not matter, so each pair is a 2-combination of the 6 patients:

$${}_6C_2 = \frac{6!}{2!4!} = \frac{6 \cdot 5}{2 \cdot 1} = 15.$$

Listing the pairs confirms this:

AB	AC	AD	AE	AF
	BC	BD	BE	BF
		CD	CE	CF
			DE	DF
				EF

Check Your Understanding

A hospital's crash cart has room for 5 of the 12 emergency medications on the approved list. How many different selections of 5 medications are possible?

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A hospital's crash cart has room for 5 of the 12 emergency medications on the approved list. How many different selections of 5 medications are possible?

Solution: Order does not matter, so each selection is a 5-combination of the 12 medications. Therefore, the number of selections is

$${}_{12}C_5 = \frac{12!}{5!7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{95,040}{120} = 792.$$

Check Your Understanding

A hospital has 8 volunteers.

- (a) Three of the volunteers will staff a booth at a health fair. In how many ways can the three be chosen?
- (b) Three of the volunteers will serve as chair, secretary, and treasurer of the volunteer committee. In how many ways can these positions be filled?

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- (b) Three of the volunteers will serve as chair, secretary, and treasurer of the volunteer committee. In how many ways can these positions be filled?

Solution:

- (a) The volunteers at the booth have no distinct roles, so order does not matter:
$${}_8C_3 = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$
- (b) The three positions are different, so order matters: ${}_8P_3 = 8 \cdot 7 \cdot 6 = 336.$

Notice that $336 = 56 \cdot 3!$, just as ${}_nP_k = {}_nC_k \cdot k!$ predicts.

Counting Arguments

So far, we have two ways to count selections of k elements from a set of n elements:

Permutations (order matters)

$${}_nP_k = \frac{n!}{(n-k)!}$$

Combinations (order does not matter)

$${}_nC_k = \frac{n!}{k!(n-k)!}$$

Counting Arguments

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$${}_nC_k = \frac{n!}{k!(n-k)!}$$

The two are related by

$${}_nP_k = {}_nC_k \cdot k!.$$

Counting Arguments

Example: A unit needs a charge nurse, an assistant charge nurse, and three aides, all chosen from a pool of 11 nurses. No nurse can fill more than one position, and the three aides have no distinct roles. In how many ways can the positions be filled?

Counting Arguments

Example: A unit needs a charge nurse, an assistant charge nurse, and three aides, all chosen from a pool of 11 nurses. No nurse can fill more than one position, and the three aides have no distinct roles. In how many ways can the positions be filled?

Solution: We fill the positions in two steps, asking each time whether order matters. The charge nurse and assistant charge nurse have different roles, so order matters, and there are ${}_{11}P_2 = 11 \cdot 10 = 110$ ways to fill these two positions. That leaves 9 nurses. The aides have no distinct roles, so order does not matter, and there are ${}_9C_3 = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways to choose them.

By the Fundamental Principle of Counting, the positions can be filled in

$${}_{11}P_2 \cdot {}_9C_3 = 110 \cdot 84 = 9,240$$

ways.

Counting Arguments

We could also have chosen the aides first. Choosing 3 aides from all 11 nurses can be done in ${}_{11}C_3 = 165$ ways, and then the charge nurse and assistant charge nurse can be chosen from the remaining 8 nurses in ${}_8P_2 = 56$ ways. This gives

$${}_{11}C_3 \cdot {}_8P_2 = 165 \cdot 56 = 9,240,$$

the same answer.

The order in which we fill the positions does not change the total – but in each step, we must count only the nurses who are still available.

Counting Arguments

If two sets A and B are disjoint, nothing is counted twice when we combine them, so $|A \cup B| = |A| + |B|$. The same is true for any number of sets, as long as no two of them share an element.

The Addition Principle

If $A_1, A_2, \dots, A_k$ are sets and no two of them have any elements in common, then

$$|A_1 \cup A_2 \cup \dots \cup A_k| = |A_1| + |A_2| + \dots + |A_k|.$$

In words: if the possibilities can be split into cases that do not overlap, the total number of possibilities is the sum of the numbers of possibilities in each case. This is called **the Addition Principle**.

Counting Arguments

Example: Mr. and Mrs. Zeta want to name their baby so that its monogram (first, middle, and last initials) is in alphabetical order with no letters repeating. How many such monograms are possible?

Counting Arguments

Example: Mr. and Mrs. Zeta want to name their baby so that its monogram (first, middle, and last initials) is in alphabetical order with no letters repeating. How many such monograms are possible?

Solution: The last initial must be Z. We split into cases by the first initial. If the first initial is A, the middle initial can be any of B through Y: 24 choices. If the first initial is B, there are 23 choices for the middle initial; if it is C, there are 22; and so on, down to 1 choice when the first initial is X. These cases do not overlap, since they have different first initials, so by the Addition Principle the number of monograms is

$$24 + 23 + 22 + \dots + 3 + 2 + 1.$$

Pairing the first and last terms, the second and second-to-last, and so on gives 12 pairs that each add to 25:

$$(24 + 1) + (23 + 2) + \dots + (13 + 12) = 12 \cdot 25 = 300.$$

Counting Arguments

There is a faster way to count the Zeta monograms. Since the last initial is always Z, a monogram is determined by its first and middle initials: two different letters from A through Y, which is 25 letters in all.

Once the two letters are chosen, alphabetical order decides which one is the first initial and which one is the middle initial. So the order in which we choose them does not matter, and each monogram corresponds to a 2-combination of the 25 letters. The number of monograms is

$${}_{25}C_2 = \frac{25!}{2!(25-2)!} = \frac{25!}{2!23!} = \frac{25 \cdot 24}{2 \cdot 1} = 300,$$

the same answer.

Counting Arguments

Example: A 5-student council must be chosen out of a class of 100. The council must have a chair, a vice-chair, and three general members. Two specific students, Alice and Bob, refuse to serve together in any capacity. How many possible councils can be formed?

Counting Arguments

Example: A 5-student council must be chosen out of a class of 100. The council must have a chair, a vice-chair, and three general members. Two specific students, Alice and Bob, refuse to serve together in any capacity. How many possible councils can be formed?

Solution: We will split the possible councils into cases that do not overlap, count each case separately, and then use the Addition Principle.

Counting Arguments

Case 1: Neither Alice nor Bob is on the council.

Counting Arguments

Case 1: Neither Alice nor Bob is on the council.

Out of the remaining 98 students, a chair and a vice-chair must be chosen, and *order matters*. Hence there are $_{98}P_2$ possibilities for the chair and vice-chair.

Counting Arguments

Case 1: Neither Alice nor Bob is on the council.

Out of the remaining 98 students, a chair and a vice-chair must be chosen, and *order matters*. Hence there are ${}_{98}P_2$ possibilities for the chair and vice-chair.

There are 96 remaining students from which to choose the 3 general members, since Alice and Bob are excluded and the chair and vice-chair have already been selected. Here, *order does not matter*. Thus, there are ${}_{96}C_3$ ways to choose the 3 general members.

Counting Arguments

Case 1: Neither Alice nor Bob is on the council.

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Therefore, the total number of councils with neither Alice nor Bob is

$${}_{98}P_2 \cdot {}_{96}C_3.$$

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case a: Alice is the chair.

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case a: Alice is the chair.

There are 98 remaining students from which to choose the vice-chair, since Alice is already chair and Bob is excluded. Thus, there are ${}_{98}P_1$ possibilities for the vice-chair.

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case a: Alice is the chair.

There are 98 remaining students from which to choose the vice-chair, since Alice is already chair and Bob is excluded. Thus, there are ${}_{98}P_1$ possibilities for the vice-chair.

There are 97 remaining students from which to choose the 3 general members. Here, *order does not matter*. Thus, there are ${}_{97}C_3$ ways to choose the general members.

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case a: Alice is the chair.

There are 98 remaining students from which to choose the vice-chair, since Alice is already chair and Bob is excluded. Thus, there are ${}_{98}P_1$ possibilities for the vice-chair.

There are 97 remaining students from which to choose the 3 general members. Here, *order does not matter*. Thus, there are ${}_{97}C_3$ ways to choose the general members.

Therefore, the total number of councils with Alice as chair is

$${}_{98}P_1 \cdot {}_{97}C_3.$$

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case b: Alice is the vice-chair.

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case b: Alice is the vice-chair.

This is just like sub-case a – the only difference is that we choose the chair instead of the vice-chair. So the total number of councils with Alice as vice-chair is also

$${}_{98}P_1 \cdot {}_{97}C_3.$$

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case c: Alice is a general member.

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case c: Alice is a general member.

There are 98 remaining students from which to choose the chair and vice-chair, since Alice is a general member and Bob is excluded. Here, *order matters*. Thus, there are $_{98}P_2$ possibilities for the chair and vice-chair.

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case c: Alice is a general member.

There are 98 remaining students from which to choose the chair and vice-chair, since Alice is a general member and Bob is excluded. Here, *order matters*. Thus, there are $_{98}P_2$ possibilities for the chair and vice-chair.

There are 96 remaining students from which to choose the other 2 general members. Here, *order does not matter*. Thus, there are $_{96}C_2$ ways to choose them.

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case c: Alice is a general member.

There are 98 remaining students from which to choose the chair and vice-chair, since Alice is a general member and Bob is excluded. Here, *order matters*. Thus, there are ${}_{98}P_2$ possibilities for the chair and vice-chair.

There are 96 remaining students from which to choose the other 2 general members. Here, *order does not matter*. Thus, there are ${}_{96}C_2$ ways to choose them.

Therefore, the total number of councils with Alice as a general member is

$${}_{98}P_2 \cdot {}_{96}C_2.$$

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case a: ${}_{98}P_1 \cdot {}_{97}C_3$

Sub-case b: ${}_{98}P_1 \cdot {}_{97}C_3$

Sub-case c: ${}_{98}P_2 \cdot {}_{96}C_2$

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case a: ${}_{98}P_1 \cdot {}_{97}C_3$

Sub-case b: ${}_{98}P_1 \cdot {}_{97}C_3$

Sub-case c: ${}_{98}P_2 \cdot {}_{96}C_2$

No council is counted in more than one sub-case, since Alice has a different position in each. So, by the Addition Principle, the total number of councils with Alice but not Bob is

$$2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2.$$

Counting Arguments

Case 2: Alice is on the council, but Bob is not.

Sub-case a: ${}_{98}P_1 \cdot {}_{97}C_3$

Sub-case b: ${}_{98}P_1 \cdot {}_{97}C_3$

Sub-case c: ${}_{98}P_2 \cdot {}_{96}C_2$

No council is counted in more than one sub-case, since Alice has a different position in each. So, by the Addition Principle, the total number of councils with Alice but not Bob is

$$2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2.$$

Case 3: Bob is on the council, but Alice is not.

This is just like Case 2, with Alice and Bob switched, so there are also

$2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2$ such councils.

Counting Arguments

Case 1: ${}_{98}P_2 \cdot {}_{96}C_3$

Case 2: $2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2$

Case 3: $2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2$

Counting Arguments

Case 1: ${}_{98}P_2 \cdot {}_{96}C_3$

Case 2: $2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2$

Case 3: $2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2$

Every allowed council falls into exactly one of these three cases, so by the Addition Principle, the total number of possible councils is

$${}_{98}P_2 \cdot {}_{96}C_3 + 2(2 \cdot {}_{98}P_1 \cdot {}_{97}C_3 + {}_{98}P_2 \cdot {}_{96}C_2) = 1,502,708,480.$$

Check Your Understanding

A hospital ethics committee needs a chair and a secretary, chosen from 10 doctors, and 2 nurse representatives with no distinct roles, chosen from 7 nurses. In how many ways can the committee be formed?

Check Your Understanding

A hospital ethics committee needs a chair and a secretary, chosen from 10 doctors, and 2 nurse representatives with no distinct roles, chosen from 7 nurses. In how many ways can the committee be formed?

Solution: The chair and secretary have different roles, so order matters, and there are ${}_{10}P_2 = 10 \cdot 9 = 90$ ways to choose them. The nurse representatives have no distinct roles, so order does not matter, and there are ${}_7C_2 = \frac{7 \cdot 6}{2 \cdot 1} = 21$ ways to choose them. By the Fundamental Principle of Counting, the committee can be formed in

$${}_{10}P_2 \cdot {}_7C_2 = 90 \cdot 21 = 1,890$$

ways.

Check Your Understanding

A wound-care team of 4 will be chosen from 7 nurses and 5 physical therapists. The team members have no distinct roles, and the team must include at least 3 nurses. How many different teams are possible?

Check Your Understanding

A wound-care team of 4 will be chosen from 7 nurses and 5 physical therapists. The team members have no distinct roles, and the team must include at least 3 nurses. How many different teams are possible?

Solution: We split the teams into cases by the number of nurses.

Case 1: Exactly 3 nurses. Choose the 3 nurses from 7 and the 1 physical therapist from 5:
 ${}_7C_3 \cdot {}_5C_1 = 35 \cdot 5 = 175$ teams.

Case 2: Exactly 4 nurses. Choose all 4 team members from the 7 nurses: ${}_7C_4 = 35$ teams.

No team has both exactly 3 and exactly 4 nurses, so the cases do not overlap. By the Addition Principle, the number of teams is $175 + 35 = 210$.

Check Your Understanding

The code for a supply closet is either two different letters from A through F followed by a digit, such as CA7, or three digits with no digit repeated, such as 482. How many codes are possible?

Check Your Understanding

The code for a supply closet is either two different letters from A through F followed by a digit, such as CA7, or three digits with no digit repeated, such as 482. How many codes are possible?

Solution: We split the codes into two cases.

Case 1: Two letters, then a digit. The order of the letters matters, so there are ${}_6P_2 = 6 \cdot 5 = 30$ ways to choose them, and then 10 choices for the digit. By the Fundamental Principle of Counting, there are $30 \cdot 10 = 300$ such codes.

Case 2: Three digits. The order matters and no digit is repeated, so there are ${}_{10}P_3 = 10 \cdot 9 \cdot 8 = 720$ such codes.

A code in Case 1 starts with a letter and a code in Case 2 does not, so the cases do not overlap. By the Addition Principle, there are $300 + 720 = 1,020$ possible codes.

Counting Arguments

In the Sets section, we saw that every set is disjoint from its complement, and that together they make up the universal set: $A \cap A^c = \emptyset$ and $A \cup A^c = U$. Since A and A^c are disjoint, the Addition Principle applies, and $|A| + |A^c| = |U|$. Subtracting $|A^c|$ from both sides gives another way to count the elements of A .

Counting with Complements

If A is a subset of a universal set U , then

$$|A| = |U| - |A^c|.$$

In words: to count the possibilities we want, we can count all of the possibilities and subtract the ones we do not want. This helps when the possibilities we do not want are easier to count, as often happens in problems that ask for *at least one* of something.

Counting Arguments

Example: A team of 4 will be chosen from 7 nurses and 5 physical therapists, and the team members have no distinct roles. How many different teams include at least one physical therapist?

Counting Arguments

Example: A team of 4 will be chosen from 7 nurses and 5 physical therapists, and the team members have no distinct roles. How many different teams include at least one physical therapist?

Solution: Let U be the set of all possible teams, and let A be the set of teams with at least one physical therapist. Then A^c is the set of teams with no physical therapists, that is, teams made up entirely of nurses.

All 4 team members are chosen from the 12 people, so $|U| = {}_{12}C_4 = 495$. A team with no physical therapists has all 4 members chosen from the 7 nurses, so $|A^c| = {}_7C_4 = 35$.

Therefore, the number of teams with at least one physical therapist is

$$|A| = |U| - |A^c| = 495 - 35 = 460.$$

Counting Arguments

We could also have split the teams into cases by the number of physical therapists. These cases do not overlap, so by the Addition Principle, the number of teams is

$$\text{Exactly 1 physical therapist:} \quad {}_5C_1 \cdot {}_7C_3 = 5 \cdot 35 = 175$$

$$\text{Exactly 2 physical therapists:} \quad {}_5C_2 \cdot {}_7C_2 = 10 \cdot 21 = 210$$

$$\text{Exactly 3 physical therapists:} \quad {}_5C_3 \cdot {}_7C_1 = 10 \cdot 7 = 70$$

$$\text{Exactly 4 physical therapists:} \quad {}_5C_4 = 5$$

$$175 + 210 + 70 + 5 = 460,$$

the same answer. The complement needed one subtraction instead of four cases.

Counting Arguments

Complements also give a faster way to count the student councils from earlier. Let U be the set of all possible councils, with no restriction on Alice and Bob, and let A be the set of allowed councils. Then A^c is the set of councils that include both Alice and Bob.

Counting Arguments

Complements also give a faster way to count the student councils from earlier. Let U be the set of all possible councils, with no restriction on Alice and Bob, and let A be the set of allowed councils. Then A^c is the set of councils that include both Alice and Bob.

To count U , choose a chair and a vice-chair from all 100 students, and then the 3 general members from the remaining 98:

$$|U| = {}_{100}P_2 \cdot {}_{98}C_3 = 9,900 \cdot 152,096 = 1,505,750,400.$$

Counting Arguments

To count A^c , we fill the positions in a different order. Alice and Bob are both on the council, so first choose the other 3 members from the remaining 98 students, in ${}_{98}C_3$ ways. Then choose the chair and vice-chair from the 5 council members, in ${}_5P_2$ ways. So

$$|A^c| = {}_{98}C_3 \cdot {}_5P_2 = 152,096 \cdot 20 = 3,041,920.$$

Counting Arguments

To count A^c , we fill the positions in a different order. Alice and Bob are both on the council, so first choose the other 3 members from the remaining 98 students, in ${}_{98}C_3$ ways. Then choose the chair and vice-chair from the 5 council members, in ${}_5P_2$ ways. So

$$|A^c| = {}_{98}C_3 \cdot {}_5P_2 = 152,096 \cdot 20 = 3,041,920.$$

Therefore, the number of allowed councils is

$$|A| = |U| - |A^c| = 1,505,750,400 - 3,041,920 = 1,502,708,480,$$

the same answer as before, without splitting into cases.

Check Your Understanding

A committee of 3 will be chosen from 6 doctors and 4 nurses, and the committee members have no distinct roles. How many different committees include at least one nurse?

Check Your Understanding

A committee of 3 will be chosen from 6 doctors and 4 nurses, and the committee members have no distinct roles. How many different committees include at least one nurse?

Solution: There are ${}_{10}C_3 = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ possible committees in all. The committees with no nurses are made up entirely of doctors, and there are ${}_6C_3 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$ of these. Therefore, the number of committees with at least one nurse is

$$120 - 20 = 100.$$

Check Your Understanding

A coin is tossed 10 times. How many of the possible sequences of heads and tails contain at least one head?

Check Your Understanding

A coin is tossed 10 times. How many of the possible sequences of heads and tails contain at least one head?

Solution: By the Fundamental Principle of Counting, there are $2^{10} = 1,024$ possible sequences in all. The only sequence with no heads is the one in which every toss is tails. Therefore, the number of sequences with at least one head is

$$1,024 - 1 = 1,023.$$

Check Your Understanding

A hospital's computer system requires a 4-digit PIN, using the digits 0 through 9. Digits may be repeated. How many possible PINs have at least one repeated digit?

Check Your Understanding

A hospital's computer system requires a 4-digit PIN, using the digits 0 through 9. Digits may be repeated. How many possible PINs have at least one repeated digit?

Solution: By the Fundamental Principle of Counting, there are $10^4 = 10,000$ possible PINs in all. The complement is the set of PINs with no repeated digit. Order matters, so there are ${}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = 5,040$ of these. Therefore, the number of PINs with at least one repeated digit is

$$10,000 - 5,040 = 4,960.$$

Counting Arguments

We have seen how to count when cases are disjoint, that is, when they do not overlap. But what about when there is overlap?

Example: In a survey of 80 patients, 45 take a blood pressure medication, 32 take a cholesterol medication, and 18 take both. How many of the patients take at least one of the two medications?

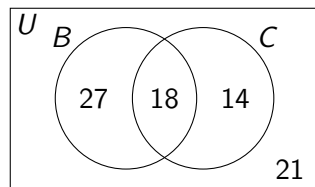
Counting Arguments

We have seen how to count when cases are disjoint, that is, when they do not overlap. But what about when there is overlap?

Example: In a survey of 80 patients, 45 take a blood pressure medication, 32 take a cholesterol medication, and 18 take both. How many of the patients take at least one of the two medications?

Solution: Let B be the set of patients who take a blood pressure medication, and let C be the set of patients who take a cholesterol medication. Adding gives $|B| + |C| = 45 + 32 = 77$, but this is too many: the 18 patients in $B \cap C$ were counted twice, once in B and once in C . Subtracting them once gives

$$|B \cup C| = 45 + 32 - 18 = 59.$$



$$27 + 18 + 14 = 59$$

Counting Arguments

The same reasoning works for any two sets: adding $|A|$ and $|B|$ counts the elements of $A \cap B$ twice, so we subtract $|A \cap B|$ once.

The Inclusion-Exclusion Principle

If A and B are sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is called **the Inclusion-Exclusion Principle**. In the survey, the patients who take neither medication are the elements of the complement $(B \cup C)^c$, so there are $80 - 59 = 21$ of them.

If A and B are disjoint, then $|A \cap B| = 0$, and the Inclusion-Exclusion Principle becomes the Addition Principle.

Counting Arguments

Example: A team of 4 will be chosen from 7 nurses and 5 physical therapists, and the team members have no distinct roles. Two of the nurses are named Ana and Ben. How many different teams include Ana or Ben?

Counting Arguments

Example: A team of 4 will be chosen from 7 nurses and 5 physical therapists, and the team members have no distinct roles. Two of the nurses are named Ana and Ben. How many different teams include Ana or Ben?

Solution: Let A be the set of teams that include Ana, and let B be the set of teams that include Ben. A team in A is decided by choosing the other 3 members from the remaining 11 people, so $|A| = {}_{11}C_3 = 165$. In the same way, $|B| = 165$. A team in $A \cap B$ is decided by choosing the other 2 members from the remaining 10 people, so $|A \cap B| = {}_{10}C_2 = 45$. By the Inclusion-Exclusion Principle,

$$|A \cup B| = |A| + |B| - |A \cap B| = 165 + 165 - 45 = 285.$$

Counting Arguments

We can check this answer using complements. The teams that include neither Ana nor Ben make up the complement $(A \cup B)^c$. Each of these teams has all 4 members chosen from the other 10 people, so

$$|(A \cup B)^c| = {}_{10}C_4 = 210.$$

There are ${}_{12}C_4 = 495$ possible teams in all, so

$$|A \cup B| = 495 - 210 = 285,$$

the same answer.

Check Your Understanding

In a class of 30 nursing students, 18 are taking anatomy, 12 are taking chemistry, and 7 are taking both.

- (a) How many of the students are taking at least one of the two courses?
- (b) How many of the students are taking neither course?

Check Your Understanding

In a class of 30 nursing students, 18 are taking anatomy, 12 are taking chemistry, and 7 are taking both.

- (a) How many of the students are taking at least one of the two courses?
- (b) How many of the students are taking neither course?

Solution: Let A be the set of students taking anatomy, and let C be the set of students taking chemistry.

- (a) By the Inclusion-Exclusion Principle, $|A \cup C| = 18 + 12 - 7 = 23$.
- (b) The students taking neither course make up $(A \cup C)^c$, so there are $30 - 23 = 7$ of them.

Check Your Understanding

How many of the whole numbers from 1 to 100 are divisible by 2 or by 5?

Check Your Understanding

How many of the whole numbers from 1 to 100 are divisible by 2 or by 5?

Solution: Let A be the set of numbers from 1 to 100 that are divisible by 2, and let B be the set of those that are divisible by 5. Then $A = \{2, 4, \dots, 100\}$, so $|A| = 50$, and $B = \{5, 10, \dots, 100\}$, so $|B| = 20$.

A number is divisible by both 2 and 5 exactly when it is divisible by 10, so $A \cap B = \{10, 20, \dots, 100\}$ and $|A \cap B| = 10$. By the Inclusion-Exclusion Principle,

$$|A \cup B| = 50 + 20 - 10 = 60.$$

Check Your Understanding

A clinic has 10 nurses, including Ana and Ben. A committee of 3 of the nurses will be chosen, and the committee members have no distinct roles. How many different committees include Ana or Ben? Find the answer in two ways: using the Inclusion-Exclusion Principle, and using complements.

Check Your Understanding

A clinic has 10 nurses, including Ana and Ben. A committee of 3 of the nurses will be chosen, and the committee members have no distinct roles. How many different committees include Ana or Ben? Find the answer in two ways: using the Inclusion-Exclusion Principle, and using complements.

Solution: Let A be the set of committees that include Ana, and let B be the set of committees that include Ben. Then $|A| = {}_9C_2 = 36$, $|B| = 36$, and $|A \cap B| = {}_8C_1 = 8$, so by the Inclusion-Exclusion Principle,

$$|A \cup B| = 36 + 36 - 8 = 64.$$

Using complements instead: there are ${}_{10}C_3 = 120$ committees in all, and ${}_8C_3 = 56$ of them include neither Ana nor Ben, so $|A \cup B| = 120 - 56 = 64$.

Counting

Tool	Use it when	Formula
Fundamental Principle of Counting	a task is done in steps	multiply the numbers of ways to do the steps
Factorials	arranging all n elements in order	$n! = n \cdot (n-1) \cdots 2 \cdot 1$
Permutations	choosing k of n elements, order matters	${}_nP_k = \frac{n!}{(n-k)!}$
Combinations	choosing k of n elements, order does not matter	${}_nC_k = \frac{n!}{k!(n-k)!}$
The Addition Principle	the cases do not overlap	$ A_1 \cup \dots \cup A_k = A_1 + \dots + A_k $
Complements	what we do not want is easier to count	$ A = U - A^c $
The Inclusion-Exclusion Principle	two sets overlap	$ A \cup B = A + B - A \cap B $